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# COMPUTABILITY, HOMOTOPY, AND TWISTED CARTESIAN PRODUCTS

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ABSTRACT. We prove that finite presentations of $\pi_n X$ are effectively computable, when $X$ is a connected, effectively locally finitely dominated nilpotent complex. The relationship between the solvability of the homotopy problem in recursive Kan complexes and the word problem in homotopy groups plays a role.

## INTRODUCTION

In 1957, E. H. Brown, Jr. [1] produced an algorithm which computed finite presentations of the homotopy groups of simply connected spaces. This author extended these results in [5], showing that the homotopy groups of a locally finite, connected, nilpotent complex are recursively computable; that is, that there is an effective procedure yielding recursively enumerable presentations of $\pi_n X$, $1 \leq n$. (Here locally finite is used to mean a complex containing finitely many simplices in each dimension.)

In this paper we strengthen these results in two ways: first, we show that the Postnikov system contains all the information needed to obtain a finite presentation of $\pi_n X$, and secondly, we show that, since locally finitely dominated complexes are homotopy equivalent to locally finite complexes (see Wall, [4]), if we adopt a suitable definition of "effectively locally finitely dominated", $\pi_n X$ is finitely computable when $X$ is a connected, nilpotent, effectively locally finitely dominated complex.

The finite computability of the presentation rests on the techniques of combinatorial group theory for presenting extensions of groups, applied to the Postnikov system which is viewed here as an "iterated principal twisted cartesian product" set within the framework of "recursive simplicial objects" developed in [5].

Throughout the paper we will work in the category of connected, pointed simplicial sets.

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## 1. Effective presentability of groups

**Proposition 1.1.** *Let $A$ and $B$ be finitely presented groups, and let $G$ be a group with an r.e. presentation. Assume that each presentation comes equipped with an algorithm for solving the word problem. Let $1 \to A \to G \to B \to 1$ be a given extension of groups in which the homomorphisms are recursive with respect to the given presentations. Then $G$ is effectively finitely presentable.*

*Proof.* Using the techniques given in Johnson [2] it is not hard to establish the result.

**Proposition 1.2.** *Let $(X, \phi)$ be a recursive Kan pair. Then $\pi_n X$ has an r.e. presentation.*

*Proof.* The generating set is certainly countable. Then enumerate the set of relators to obtain an r.e. multiplication table of the group.

**Definition 1.3.** Let $X$ be a complex. The homotopy problem for $X$ is the problem of deciding, for any two $n$-simplices $x$ and $y$, $n \geq 0$, such that $d_i x = \phi = d_i y$, $0 \leq i \leq n$, whether $x \sim y$.

**Proposition 1.4.** *Let $X$ be a recursive Kan complex for which the homotopy problem is decidable. Then $\pi_n X$ has an r.e. presentation with solvable word problem.*

*Proof.* The proof is an easy exercise in induction on the length of the word and is left to the reader.

**Definition 1.5.** An *iterated principal twisted cartesian product* (PTCP) $F \times_\mu B$ of rank $n$ may be defined inductively as follows:

(1) $n = 1$: $F \times_\mu B$ is a (P)TCP.
(2) An iterated (P)TCP of rank $n$ is a (P)TCP whose base is an iterated (P)TCP of rank $n - 1$.

Note that the structural group of the TCP may change with each iteration. A *recursive iterated* PTCP is an iterated PTCP in which all the complexes are recursive and all the twisting functions are recursive functions.

**Proposition 1.6.** *Let $G$ be a group with an r.e. presentation with solvable word problem. Let $g_0, \dots, g_{n+1}$ be a collection of $n+2$ $n$-simplices of $K(G, n)$. Then it is decidable whether there exists $g \in K(G, n)^{n+1}$ such that $d_i g = g_i$ for all $i$, and if $g$ exists it is both unique and effectively computable.*

*Proof.* This is easily accomplished by induction, using the Bar construction of $K(G, n)$.

**Proposition 1.7.** *Let $K$ be a recursive iterated PTCP of $K(G_i, n)$'s, $1 \leq i \leq k$, in which the twisting functions all preserve the base point. Then it is possible to decide whether two $n$-simplices are homotopic.*

*Proof.* The proof proceeds by induction in $i$. If $i = 1$ then it is easy to see that $x \sim y$ if and only if $x = y$.

For the induction step assume that the theorem is true for recursive iterated PTCP's $Y$ of rank less than $p$, and assume further that if $z \in Y^{n+1}$ is a homotopy between $x$ and $y$ that $z$ is unique. Let $X = K(G, n) \times_\tau Y$ and let $x_0 = (g_0, y_0)$, $x_1 = (g_1, y_1)$.

Now $x_0 \sim x_1$ if and only if there exists a pair $(h, z) \in K(G, n)^{n+1} \times Y^{n+1}$ such that $d_i(h, z) = \phi$, $i < n$; $d_n z = (g_0, y_0)$; and $d_{n+1} z = (g_1, y_1)$. These conditions are met if and only if $z$ is a homotopy from $y_0$ to $y_1$ and $d_0 h = \tau(z)^{-1}$, $d_i h = \phi$, $1 \le i < n$; $d_n h = g_0$; and $d_{n+1} h = g_1$.

The decidability of the existence of $z$ is assured by the induction hypotheses, and if $z$ exists it must be unique. This determines explicitly all the faces of $h$, and the rest of the proposition follows from Proposition 1.6.

**Proposition 1.8.** *Let $X = K(G_k; n_k) \times_\tau \cdots \times_\tau K(G_1; n_1)$ be a recursive iterated PTCP of $K(G_i; n_i)$'s, where $1 \le n_i \le n_{i+1} \le n$, in which every twisting function is base-point preserving. Then, for every n-simplex $x$ satisfying $d_i x = \phi$ for all $i$, $0 \le i \le n$, the homotopy problem is decidable.*

*Proof*. Let $x$ and $y$ be $n$-simplices of $X$ such that $d_i x = d_i y = \phi$ for all $0 \le i \le n$. The component of $x$ which lies in $K(G_i; p)$ for $p < n$ must be the identity element. This follows from Proposition 1.2 and the fact that the twisting functions preserve the identity elements. Now break $X$ into a TCP $A \times_\tau B$, where $B$ is the iterated PTCP of $K(G_i; n_i)$'s, $1 \le n_i < n$. Then if $X = (a_0, b_0)$ and $y = (a_1, b_1)$, $b_i = e$ for $i = 0, 1$; so $x \sim y$ if and only if $a_0 \sim a_1$, which is decidable according to Proposition 1.7.

## 2. Computability of $\pi_n X$

The example of an iterated PTCP which primarily concerns us is that of the Postnikov system of a nilpotent complex. By the $n$th stage $P_{n,i}(X)$ of the Postnikov system of a nilpotent complex $X$, we mean the iterated PTCP $P_{n,i}(X) = (3)$:

(3) $$K(G_{n,i}; n) \times_{\tau(n,i)} [\cdots [K(G_{1,2}; 1) \times_{\tau(1,2)} K(G_{1,1}; 1)]] \cdots ].$$

Here $\tau(p, q)$ is the composite of $A_{p,q}: P_{p,q-1} \to K(G_{p,q}; p+1)$ with the canonical twisting function $\tau: K(G_{p,q}; p+1) \to K(G_{p,q}; p)$, and the groups $G_{n,i}$ are the $i$th successive quotients of the lower central series of the action of $\pi_1 X$ on $\pi_n X$. Since $\tau(p, q)$ is pointed it does not preserve the identity element.

The following theorem is proved in Weld [5], although it is stated here in a slightly different form, using the language of iterated PTCPs.

**Theorem 2.1.** *Let $X$ be a connected locally finite nilpotent complex. Then the nth stage $P_{n,i}$ of the Postnikov system of $X$ is effectively constructible as an iterated PTCP of $K(G_{n,i}; n)$'s in which the groups $G_{n,i}$ are all finitely generated abelian groups.*

**Definition 2.2.** We will say that $Y$ *effectively dominates* $X$, if $X$ and $Y$ are recursive and there exist recursive maps $r: Y \to X$ and $j: X \to Y$ such that $r$ composed with $j$ is homotopic to the identity map on $X$.

Recall that a group $G$ is said to be a *retract* of a group $H$ if there are homomorphisms $j: G \to H$ and $r: H \to G$ such that $rj = 1_G$. The following lemma, which is stated here in constructive form, is due originally to Wall, [4].

**Lemma 2.3.** *Let $H$ and $G$ be groups. Let $H$ be a retract of $G$. Suppose $G$ is finitely presented, $H$ has an r.e. presentation, and that the maps $r: G \to H$ and $j: H \to G$ such that $rj = 1_H$ are effectively computable in terms of the generators of the given presentations. Then there is an effective procedure for determining a finite presentation of $H$, in which the generating set is precisely the set of generators for the presentation of $G$.*

**Theorem 2.4.** *Let $X$ be effectively dominated by a locally finite complex $Y$. Suppose $X$ is a nilpotent complex. Then the conclusions of Theorem* 2.1 *hold for $X$.*

*Proof.* Theorem 2.4 is proved by modifying the proof of Theorem 2.1, applying, as needed, Lemma 2.3. We leave the details to the specialist.

**Theorem 2.5.** *Let $X$ be a connected nilpotent effectively locally finitely dominated complex. Then, for $n \geq 1$, $\pi_n x$ is an effectively finitely presentable group with solvable word problem.*

*Proof.* Since we can construct the Postnikov system of $X$ as a tower of recursive Kan complexes, it is clear that there is an r.e. presentation of $\pi_n X$. Furthermore Proposition 1.8 guarantees that the homotopy problem is solvable for those $n$-simplices which represent elements of $\pi_n X$. Therefore $X$ comes equipped with an algorithm for solving the word problem in the r.e. presentation of $\pi_n X$. In fact this is true of $\pi_n P_{n,i}$ for all pairs $n, i$. Then, to see that $\pi_1 X$ must be finitely presentable, we apply Proposition 1.1: We need only to verify that $\pi_1(X)$ is an extension of finitely generated abelian groups. This is easily seen by inducting up the Postnikov tower, using the long exact sequence of a principal fibration.

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