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CONGRUENCE AND DIMENSION OF NON-SEPARABLE METRIC SPACES

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Dedicated to Professor Yukihiro Kodama on his 60th birthday

ABSTRACT. In this paper, we prove that, if a metrizable space $\mathbf{X}$ has an admissible metric such that $\mathbf{X}$ has no two distinct congruent subsets of cardinality 3, then $\operatorname{ind}\mathbf{X} \leq 1$. We also show that if a non-empty metrizable space $\mathbf{X}$ has an admissible star-rigid metric, then $\operatorname{ind}\mathbf{X} = 0$. The latter answers a question of L. Janos and H. Martin [3].

In the present note, we investigate a relation between certain special metrics and dimensions of non-separable metrizable spaces. For a metric space (X, d), a point $x \in \mathbf{X}$ and a positive real number ε, we denote $U(x;\varepsilon) = \{y \in \mathbf{X}: d(x,y) < \varepsilon\}$ and $H(x;\varepsilon) = \{y \in \mathbf{X}: d(x,y) = \varepsilon\}$. Recall from [2] that two subsets $\mathbf{A}$ and $\mathbf{B}$ of a metric space $(\mathbf{X}, d)$ are called *congruent* if there exists a bijection f from $\mathbf{A}$ onto $\mathbf{B}$ such that $d(a,b) = d(f(a), f(b))$ for every a, $b \in \mathbf{A}$. In [1], Janos proved that a non-empty separable metrizable space $\mathbf{X}$ is zero-dimensional if, and only if, $\mathbf{X}$ has an admissible metric d such that $\mathbf{X}$ contains no two distinct congruent subsets of cardinality 2. Recently, he explored the case where the cardinality of the subsets is 3 in [2]. In particular, he showed that a locally compact, separable metrizable space $\mathbf{X}$ is at most one-dimensional if $\mathbf{X}$ has an admissible metric d satisfies the following condition:

($*$) $\mathbf{X}$ *has no two distinct subsets of cardinality* 3 *that are congruent relative to* d.

On the other hand, there exists a non-separable metric space $(\mathbf{X}, d)$ that satisfies the condition $(*)$ (see the example later in this paper). Thus, the following theorem improves the result of Janos.

Theorem 1. *If the metrizable space* $\mathbf{X}$ *has an admissible metric satisfying the condition* $(*)$, *then* $\operatorname{ind}\mathbf{X} \leq 1$.

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Proof. Let d be an admissible metric on $\mathbf{X}$ that satisfies the condition $(*)$. Let $x \in \mathbf{X}$ and $\varepsilon > 0$. It suffices to show that $\operatorname{ind} H(x;\varepsilon) \leq 0$. To show it, let $y \in H(x;\varepsilon)$ and $\delta > 0$. By the condition $(*)$, it follows that the cardinality of the set $H(x;\varepsilon) \cap H(y;\delta)$ is at most one. First, we suppose that $H(x;\varepsilon) \cap H(y;\delta) = \varnothing$. It is obvious that $V(y) = H(x;\varepsilon) \cap U(y;\delta)$ is an open-and-closed neighborhood of y in $H(x;\varepsilon)$.

Next, we suppose that $H(x;\varepsilon) \cap H(y;\delta) = \{z\}$. We put $G = \{w \in \mathbf{X}: d(y,w) < d(z,w)\}$ and $V(y) = G \cap H(x;\varepsilon) \cap U(y;\delta)$. To show that the open neighborhood $V(y)$ of y in $H(x;\varepsilon)$ is closed in $H(x;\varepsilon)$, let $w \in H(x;\varepsilon) - V(y)$. If $w \notin G$, the condition $(*)$ implies $d(y,w) > d(z,w)$. Then let $\delta' = d(y,w) - d(z,w) > 0$. It is easy to show that $G \cap U(w;\delta'/2) = \varnothing$. Hence $U(w,\delta'/2) \cap V(y) = \varnothing$. If $w \in G$, then $d(y,w) \geq \delta$.

Suppose that $d(y,w) = \delta$. Then $w \in H(x;\varepsilon) \cap H(y;\delta)$. On the other hand, $z \neq w$ and $z \in H(x;\varepsilon) \cap H(y;\delta)$. Hence $|H(x;\varepsilon) \cap H(y;\delta)| \geq 2$; this is a contradiction. Therefore, $d(y,w) > \delta$.

We put $\delta'' = d(y,w) - \delta > 0$. It follows that $U(w;\delta'') \cap U(y;\delta) = \varnothing$. Hence $U(w;\delta'') \cap V(y) = \varnothing$. Thus $V(y)$ is closed in $H(x;\varepsilon)$. This completes the proof.

The following example shows that the condition $(*)$ does not imply the separability of spaces.

Example. Well-order the set of all real numbers in the half-open interval $(0,1]$ as $\{t_\alpha: \alpha \in A\}$, where A denotes the set of all ordinal numbers less than the cardinal of the continuum c. Let $\mathbf{S}(A)$ denote the star-space with an index set A (see [4, p. 111] for the definition of the star-spaces). Let ρ be the metric on $\mathbf{S}(A)$ defined as

$$\rho((x,\alpha),(y,\beta)) = \begin{cases} |x-y|, & \text{if } \alpha = \beta, \\ x+y, & \text{if } \alpha \neq \beta, \end{cases}$$

for each (x,α), $(y,\beta) \in \mathbf{S}(A)$.

Let $\mathbf{X} = \{(t_\alpha,\alpha): \alpha \in A\}$ be the subspace of $\mathbf{S}(A)$. It is easy to show that the metric space $(\mathbf{X},\rho)$ satisfies the condition $(*)$, but the weight of $(\mathbf{X},\rho)$ is equal to c.

In [3], Janos and Martin studied the relations between the star-rigid metric and zero-dimensionality of separable metrizable spaces. Recall from [3] that a metric d on a space $\mathbf{X}$ is called *star-rigid* if, for every points x, y and z of $\mathbf{X}$ with $y \neq z$, $d(x,y) \neq d(x,z)$. Janos and Martin proved that for a non-empty separable metrizable space $\mathbf{X}$, $\mathbf{X}$ is zero-dimensional if, and only if, $\mathbf{X}$ has an admissible totally-bounded star-rigid metric, and they asked whether for every non-empty metrizable (not necessarily separable) space $\mathbf{X}$, $\operatorname{ind}\mathbf{X} = 0$ holds, if $\mathbf{X}$ has an admissible star-rigid metric [3, Question 2]. An argument similar to the proof of Theorem 1 shows the following theorem which answers the question.

Theorem 2. *If a non-empty metrizable space* $\mathbf{X}$ *has an admissible star-rigid metric, then* $\operatorname{ind}\mathbf{X} = 0$.

Proof. Let d be an admissible metric on $\mathbf{X}$, $x \in \mathbf{X}$ and $\varepsilon > 0$. If $H(x;\varepsilon) = \varnothing$, then $U(x;\varepsilon)$ is an open-and-closed neighborhood of x.

Suppose that $H(x;\varepsilon) = \varnothing$. Since d is star-rigid, $|H(x;\varepsilon)| = 1$. Let $\{z\} = H(x;\varepsilon)$ and $G = \{w \in \mathbf{X}: d(x,w) < d(z,w)\}$. Since d is star-rigid, G is an open-and-closed set of $\mathbf{X}$. We put $V = U(x;\varepsilon) \cap G$. It can be seen that V is an open-and-closed neighborhood of x by an argument similar to the proof of Theorem 1. Hence $\operatorname{ind}\mathbf{X} = 0$.

Remark. It is obvious that $\operatorname{ind}\mathbf{X} = 0$ for the metric space $(\mathbf{X}, \rho)$ mentioned in the example. Therefore, we can ask the following question: Does there exist a non-separable metric space $(\mathbf{X}, d)$ such that $\operatorname{ind}\mathbf{X} = 1$ and the condition $(*)$ is satisfied?

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