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A UNIQUENESS CONDITION FOR FINITE MEASURES

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ABSTRACT. Let μ and μ' be two finite measures on the same measurable space which have the property: $\mu(A) = \mu(B)$ implies that $\mu'(A) = \mu'(B)$. If the range of μ is an interval, then there is a constant α such that $\mu' = \alpha\mu$. This extends earlier results of Leth and Malitz on purely atomic measures.

1. INTRODUCTION

Recently Malitz [6] proved the following result.

Theorem 1. *Let* $\langle a_n \rangle$ *and* $\langle a_n' \rangle$ *be two sequences of real numbers such that*

(i) $a_n \to 0$ *and* $a_n' \to 0$,
(ii) $0 < a_{n+1} \le a_n$ *and* $0 < a_n' \le a_{n+1}'$ *for all* n,
(iii) $a_n \le \sum_{j>n} a_j$ *and* $a_n' \le \sum_{j>n} a_j'$ *for all* n,
(iv) $\sum_{i\in I} a_i = \sum_{j\in J} a_j$ *iff* $\sum_{i\in I} a_i' = \sum_{j\in J} a_j'$.

Then there is a constant α *such that* $a_n' = \alpha a_n$ *for all* n.

This theorem is a strengthening of an earlier result of Leth [4]. In Leth's theorem (iv) is replaced by

$$\sum_{i\in I} a_i \le \sum_{j\in J} a_j \text{ iff } \sum_{i\in I} a_i' \le \sum_{j\in J} a_j'.$$

Theorem 1 can be interpreted as a result on purely atomic measures, and in fact its origins [2] are in the study of purely atomic measures. (See also [1] and [7] for related results on nonatomic measures.) The purpose of this paper is to extend this result to arbitrary finite measures. The main result is

Theorem 2. *Let* μ *and* μ' *be two finite measures on the same measurable space which have the property:*

$$(*) \qquad \mu(A) = \mu(B) \text{ implies } \mu'(A) = \mu'(B).$$

If the range of μ *is an interval, then there is a constant* α *such that* $\mu' = \alpha\mu$.

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The proof of this theorem uses Theorem 1 and will be given in §4. Note that if we take μ in Theorem 2 to be a purely atomic measure we obtain the following corollary, which is analogous to Theorem 1.

Corollary 1. *Let $\langle a_n \rangle$ and $\langle a'_n \rangle$ be two sequences of real numbers such that*

(i′) $\sum a_n$ *and* $\sum a'_n$ *converge,*

(ii′) $0 < a_{n+1} \le a_n$ *and* $0 \le a'_n$ *for all* n,

(iii′) $a_n \le \sum_{j>n} a_j$ *for all* n,

(iv′) $\sum_{i \in I} a_i = \sum_{j \in J} a_j$ *implies that* $\sum_{i \in I} a'_i = \sum_{j \in J} a'_j$.

Then there is a constant α *such that* $a'_n = \alpha a_n$ *for all* n.

Comparing Corollary 1 with Theorem 1, note that (i′) is a stronger condition than (i). The reason is, of course, that Theorem 2 applies only to finite measures. However, (ii′), (iii′), and (iv′) are weaker than (ii), (iii), and (iv) respectively.

To see that Corollary 1 follows from Theorem 2, note that (iii′) is a necessary and sufficient condition for the range of the purely atomic measure corresponding to the sequence $\langle a_n \rangle$ to be an interval. This is included in Proposition 1 of the next section.

2. Purely atomic case

In this section $\langle a_n \rangle$ will be a sequence of positive real numbers with $\sum a_n = s$ and the terms ordered so that $0 < a_n \le a_{n+1}$ for all n. We also let $r_n = \sum_{k>n} a_k$ and denote the complement of a set A by A^c.

The following result can be found in [4] and [6].

Proposition 1. *For every* $x \in [0, s]$ *there is a set* I_x *such that* $x = \sum_{n \in I_x} a_n$ *iff* $a_n \le r_n$ *for all* n. *In this case, if* $x \in (0, s]$, I_x *can be chosen so that* I_x *is infinite. Also, if* $x \in [0, s)$, I_x *can be chosen so that* I_x^c *is infinite.*

Remark 1. If $a_n \le r_n$ for all n, then for each n there is a set J_n such that $a_n = \sum_{k \in J_n} a_k$, and it is easy to see that we can choose J_n so that $\min J_n = n+1$.

In what follows we will assume that $a_n \le r_n$ for all n and that $\langle a'_n \rangle$ is another sequence of real numbers for which $a'_n \ge 0$ for all n and $\sum a'_n$ converges. Let $s' = \sum a'_n$ and $r'_n = \sum_{i>n} a'_i$ for all n. We shall also assume that $\langle a_n \rangle$ and $\langle a'_n \rangle$ satisfy

$$(**) \qquad \sum_{i \in I} a_i = \sum_{j \in J} a_j \text{ implies } \sum_{i \in I} a'_i = \sum_{j \in J} a'_j .$$

Remark 2. Referring to Remark 1, we have from $(**)$ $a'_n = \sum_{k \in J_n} a'_k$ with $\min J_n = n+1$. Hence we see that $a'_{n+1} \le a'_n$ and $a'_n \le r'_n$.

Proposition 2. *If* $a'_m = 0$ *for some* m, *then* $a'_n = 0$ *for all* n.

Proof. Assume that $a'_m = 0$. By Remark 2, $a'_{m+1} \le a'_m = 0$. Hence $a'_{m+1} = 0$ and by an easy induction $a'_n = 0$ for $n > m$. Also by Remark 1, there is a set J_{m-1} such that $a'_{m-1} = \sum_{k\in J_{m-1}} a'_k$ with $\min J_{m-1} = m$. Hence we have $a'_{m-1} = 0$. Therefore, by backwards induction, $a_n = 0$ for $n < m$.

In light of Proposition 2 we shall assume $a'_n > 0$ for all n. (If $a'_n = 0$ for some n, take $\alpha = 0$ in Theorem 3.)

We now define a function $f\colon [0,s] \to [0,s']$ by $f(x) = \sum_{n\in I_x} a'_n$ where I_x is as given in Proposition 1. Note that $f(x)$ is independent of the choice of I_x by (**). Also note that $f(s-x) = s' - f(x)$ for all $x \in [0,s]$ since I_{s-x} can be taken to be I_x^c. We now prove some results about this function f.

Proposition 3. *f is continuous on* $[0,s]$.

Proof. Let $\varepsilon > 0$ be given and let $x \in [0,s)$. Then by Proposition 1 we can choose a set I_x so that I_x^c is infinite and $x = \sum_{n\in I_x} a_n$. Choose N so that $\sum_{n\geq N} a'_n < \varepsilon/2$ and let $\delta = \sum_{\substack{n\in I_x^c\\ n>N}} a_n$. Note that $\delta > 0$ since I_n^c is infinite. Then, if $x < y < x+\delta$, there is a set $J \subseteq \{n\colon n \geq N\}$ for which $y = \sum_{\substack{n\in I_x\\ n<N}} a_n + \sum_{n\in J} a_n$. Letting $x_0 = \sum_{\substack{n\in I_x\\ n<N}} a_n$ we have

$$\begin{aligned} |f(y) - f(x)| &\leq |f(y) - f(x_0)| + |f(x) - f(x_0)| \\ &= \sum_{n\in J} a'_n + \sum_{\substack{n\in I_x\\ n>N}} a'_n < \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon . \end{aligned}$$

Therefore f is right continuous at x. Since $f(s-x) = s' - f(x)$, f is also left continuous at x for $x \in (0,s]$. Hence f is continuous on $[0,s]$.

Proposition 4. *f is strictly increasing on* $[0,s]$.

Proof. Assume that f is not strictly increasing on $[0,s]$. Then, since f is continuous, f has an interior local maximum, say at z. Now I_z may be chosen so that I_z^c is infinite and $z = \sum_{n\in I_z} a_n$. Let $\langle a_{n_k}\rangle$ be a subsequence of $\langle a_n\rangle$ for which $n_k \in I_z^c$ for all k. Then $f(z + a_{n_k}) = f(z) + a'_{n_k} > f(z)$ for all k. Hence there are points arbitrarily close to z with larger function values than at z. This contradicts the fact that f has a local maximum at z.

The following is an immediate consequence of Proposition 4.

Corollary 2. *f is one-to-one. Hence $\sum_{i\in I} a'_n = \sum_{j\in J} a'_j$ implies that $\sum_{i\in I} a_n = \sum_{j\in J} a_j$.*

Theorem 3. *Let μ and μ' be two finite measures on the same measurable space for which $\mu(A) = \mu(B)$ implies that $\mu'(A) = \mu'(B)$. If the range of μ is an interval and μ is purely atomic, then $\mu' = \alpha\mu$ for some constant α.*

Proof. Since the range of μ is an interval, μ has a countable infinity of atoms. Denote the sequence of μ measures of the atoms by $\langle a_n\rangle$ and without loss of generality assume that $a_{n+1} \leq a_n$ for all n. Denote by $\langle a'_n\rangle$ the μ' measures of the atoms of μ and note that condition $(**)$ is satisfied. If $a'_m = 0$ for some m, $a'_n = 0$ for all n by Proposition 2 and we take $\alpha = 0$. Now assume that $a'_n > 0$ for all n. Then by Remark 2, Corollary 2, and construction, all the conditions of Theorem 1 are satisfied. Hence there is a constant α such that $a'_n = \alpha a_n$ for all n and therefore $\mu' = \alpha\mu$.

3. Nonatomic case

In this section we use the following two results. The first can be found in [5, p. 100].

Proposition 5. *If μ is a nonatomic measure on a set X, then the range of μ is $[0, \mu(X)]$.*

Proposition 6. *If f is a function defined on $[0, a]$ which is continuous at 0 and additive (i.e., $x, y, x+y \in [0, a]$ implies $f(x+y) = f(x) + f(y)$), then there is a constant α such that $f(x) = \alpha x$ for all $x \in [0, a]$.*

Proof. This result is well known in the case where the domain of f is $[0, \infty)$ but appears to be less well known as stated. For this reason we will give a sketch of the proof. First note that by replacing $f(x)$ by $f(x/a)$ we may assume that $a = 1$. Next by an easy induction one can show that if $0 \le x \le 1/n$, then $f(nx) = nf(x)$. In particular $f(1) = nf(1/n)$. Hence $f(1/n) = f(1)/n$ and therefore $f(m/n) = mf(1/n) = f(1)m/n$ if $0 \le m \le n$. It is easy to show that additivity and continuity at 0 implies continuity on $[0, 1]$. Finally, continuity and $f(x) = f(1)x$ for all rational x in $[0, 1]$ implies $f(x) = \alpha x$ for all $x \in [0, 1]$ with $\alpha = f(1)$.

We shall use the following result to prove the main theorem of this section.

Lemma 1. *Assume that μ and μ' are finite measures on the measurable space $(\Omega, \mathscr{B})$ which satisfy $(*)$. If μ is nonatomic and $\mu'(\Omega) \neq 0$, then for any $\varepsilon > 0$ there is a set $A \in \mathscr{B}$ such that $0 < \mu'(A) < \varepsilon$.*

Proof. Given $\varepsilon > 0$ choose N such that $\mu'(\Omega) < N\varepsilon$. Using Proposition 5 we can find disjoint sets $A_1, A_2, \ldots, A_N \in \mathscr{B}$ such that $\Omega = \bigcup_{n=1}^{N} A_n$ and $\mu(A_1) = \mu(A_2) = \cdots = \mu(A_N)$. Hence by $(*)$,

$$\mu'(A_1) = \mu'(A_2) = \cdots = \mu'(A_N) = \mu'(\Omega)/N < \varepsilon.$$

Taking $A = A_1$, the lemma is proved.

Theorem 4. *Let μ and μ' be two finite measures on the measurable space $(\Omega, \mathscr{B})$ which satisfy*

$$(*) \qquad \mu(A) = \mu(B) \text{ implies } \mu'(A) = \mu'(B).$$

If μ is nonatomic, then $\mu' = \alpha\mu$ for some constant α.

Proof. If $\mu'(\Omega) = 0$, take $\alpha = 0$. Now assume $\mu'(\Omega) > 0$. Define a function $f\colon [0, \mu(\Omega)] \to [0, \mu'(\Omega)]$ as follows. If $x \in [0, \mu(\Omega)]$ there is, by Proposition

5, a set $A_x \in \mathscr{B}$ such that $\mu(A_x) = x$. Let $f(x) = \mu'(A_x)$. By $(*)$, $f(x)$ is independent of the choice of A_x.

We will first show that f is continuous at 0. Let $\varepsilon > 0$ be given. By Lemma 1, there is a set $A \in \mathscr{B}$ such that $0 < \mu'(A) < \varepsilon$. Let $\delta = \mu(A)$. Note that $\delta > 0$ since if $\mu(A) = 0 = \mu(\phi)$, then $\mu'(A) = \mu'(\phi) = 0$. Now if $0 \leq x < \delta$, then by Proposition 5 there is a set $A_x \subseteq A$ such that $x = \mu(A_x)$. Hence

$$|f(x) - f(0)| = f(x) = \mu'(A_x) \leq \mu'(A) < \varepsilon$$

and we see that f is continuous at 0.

Secondly we show that f is additive. Assume that $x, y, x+y \in [0, \mu(\Omega)]$. Then by Proposition 5 there is a set $A_x \in \mathscr{B}$ such that $x = \mu(A_x)$ and a set $A_y \in \mathscr{B}$ such that $A_y \subseteq A_x^c$ and $y = \mu(A_y)$. Then

$$\begin{aligned} f(x+y) &= f(\mu(A_x \cup A_y) \\ &= \mu'(A_x \cup A_y) \\ &= \mu'(A_x) + \mu'(A_y) \\ &= f(x) + f(y). \end{aligned}$$

Now by Proposition 6, there is a constant α such that $f(x) = \alpha x$ for all $x \in [0, \mu(\Omega)]$. Hence $\mu' = \alpha\mu$.

4. General case

We now give the proof of Theorem 2. In Theorem 3 we addressed the case where μ is purely atomic and in Theorem 4 the case where μ is nonatomic. We now consider the mixed case. Again let $(\Omega, \mathscr{B})$ be the measurable space on which μ and μ' are defined. Let $\Omega = C \cup D$, where D is the union of all the atoms of μ. We now consider the case where $\mu(C) \neq 0$ and $\mu(D) \neq 0$. Let $\mathscr{B}_C = \{A \cap C : A \in \mathscr{B}\}$ and let μ_C and μ'_C be the restrictions of μ and μ' respectively to $\mathscr{B}_C$. Now by Theorem 4 there is a constant α such that $\mu'_C = \alpha\mu_C$. Suppose that $E \in \mathscr{B}$ and $\mu(E) \leq \mu(C)$. Then there is a set $B \in \mathscr{B}_C$ for which $\mu(E) = \mu(B)$. Hence by $(*)$, $\mu'(E) = \mu'(B) = \alpha\mu(B) = \alpha\mu(E)$. Now let $A_1, A_2, \ldots, A_m$ be those atoms of μ for which $\mu(A_i) > \mu(C)$ ordered such that $\mu(A_1) \leq \mu(A_2) \leq \cdots \leq \mu(A_m)$. Let $D' = D \sim \bigcup_{i=1}^m A_i$. By what we have just shown, for all measurable subsets of $C \cup D'$, $\mu'(E) = \alpha\mu(E)$. Now we show that there must be a measurable subset E_1 of $C \cup D'$ such that $\mu(E_1) = \mu(A_1)$. From Proposition 5, for any $x \in [0, \mu(C)]$ there is a subset E of C with $\mu(E) = x$. Now since all atoms of D' have μ measure less than or equal to $\mu(C)$, it follows that for any $x \in [0, \mu(C \cup D')]$ there is a subset E of $C \cup D'$ such that $\mu(E) = x$. Now $\mu(A_1) \leq \mu(C \cup D')$, since otherwise the interval $(\mu(C \cup D'), \mu(A_1))$ would be disjoint from the range of μ. Consequently, there is indeed a subset E_1 of $C \cup D'$ for which $\mu(E_1) = \mu(A_1)$. Hence by $(*)$, $\mu'(E_1) = \mu'(A_1)$. Therefore, $\mu'(A_1) = \mu'(E_1) = \alpha\mu(E_1) = \alpha\mu(A_1)$. Similarly there is a measurable subset B_2 of $C \cup D' \cup A_1$ such that $\mu(E_2) = \mu(A_2)$, and as before, $\mu'(A_2) = \alpha\mu(A_2)$. Continuing, we obtain, by finite induction that $\mu'(A_m) = \alpha\mu(A_m)$. Hence $\mu'(E) = \alpha\mu(E)$ for all $E \in \mathscr{B}$.

5. Questions on possible extensions

It is natural to ask whether one can extend Theorem 2 to σ-finite measures. At present the author is unable to do this for arbitrary σ-finite measures, but there are two special cases which deserve mention.

First, if μ is a σ-finite measure with a nontrivial nonatomic part, then the result of Theorem 2 holds. This can be shown with slight modifications to the proof of Theorem 2 given in the last section.

Second, if μ is a purely atomic σ-finite measure for which the μ measures of the atoms decrease to 0, then the result of Theorem 2 also holds. This can be shown using results similar to those of §2 and using Theorem 1. Recall that Theorem 1 also holds for divergent series.

It is also natural to ask whether one can relax the requirement in Theorem 2 that the range of μ be an interval. Guthrie and the author have recently shown [3] that the range of any finite measure is always one of the following:

(1) a finite set,
(2) a finite union of closed intervals,
(3) homeomorphic to the Cantor set, or
(4) homeomorphic to a set described in detail in [3].

(For our purposes it is sufficient to know that this set and its complement both contain infinitely many intervals.)

If the range of μ is a finite set it is not difficult to see that the conclusion of Theorem 2 does not hold unless the range of μ is $\{ka\colon k = 0, 1, \ldots, n\}$ for $n = 0, 1, 2, \ldots$.

The following example shows that the conclusion of Theorem 2 does not hold if the range of μ is a union of two disjoint intervals. Similar examples can be constructed where the range of μ is a union of more than two disjoint intervals.

Example. Let μ and μ' be the purely atomic measures determined by the sequences $\langle a_n \rangle$ and $\langle a'_n \rangle$ respectively, defined by $a_n = a'_n = 1/2^n$ for $n \geq 1$ and $a_0 = a$ and $a'_0 = b$ with $a > b > 1$. Then the range of μ is easily seen to be $[0, 1] \cup [a, a+1]$ and the range of μ' is $[0, 1] \cup [b, b+1]$. Hence μ' is not a multiple of μ. It is easy to check that condition $(*)$ is satisfied.

Leth [4] has given examples where the range of μ is homeomorphic to the Cantor set and the conclusion of Theorem 2 does not hold, but has also given an example where the conclusion does hold.

At present very little is known about the case where the range is a set of the fourth type.

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