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A REMARK ON λ_{2g-2}

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ABSTRACT. It is shown that if the first $2g-3$ Laplace eigenvalues on a compact Riemann surface of genus ≥ 2 are small, then λ_{2g-2} is greater than $\frac{1}{4}$.

It is known [3, 10] that the Laplacian on a compact Riemann surface M of genus $g \geq 2$ can have eigenvalues in $(0, \frac{1}{4})$, that there can be at most $4g-3$ such eigenvalues [3], and that at most $2g-3$ of these can be arbitrarily small [6, 7, 8, 11].

It is the purpose of this note to point out that if the latter are present and small enough, i.e., if $\lambda_1, \ldots, \lambda_{2g-3}$ are sufficiently small, then there are no further eigenvalues in $(0, \frac{1}{4}]$, or what is the same thing, $\lambda_{2g-2} > \frac{1}{4}$. Our approach does not shed light on the whereabouts of λ_{2g-2} when $\lambda_1, \ldots, \lambda_{2g-3}$ are present in $(0, \frac{1}{4})$ but not small (cf. the concluding remark). Peter Buser has informed me that he has obtained a similar result (unpublished).

In order to prove the result, we take a dissection of M into 3-holed spheres by a collection C of $3g-3$ simple closed geodesics in M having minimal aggregate length. This can be done in such a way that $|C|$, the sum of the lengths of the geodesics in C, is bounded by a function of g (cf. [1, 5]). It is then known that for a given g, λ_{2g-3} is close to zero if and only if $|C|$ is close to zero, and that λ_{2g-3} and $|C|$ are tightly coupled when either is small, in the sense that their ratio is contained between bounded limits [2, 6, 7, 8, 9, 11]. Now a three-holed sphere of the above type is determined up to isometry by specifying the lengths of its three holes, and a model possessing any prescribed triple of hole lengths can be created by gluing together appropriate sides of two copies of a suitable right-angled geodesic hexagon (cf. [5]). Moreover, it is proved in [4] that if the holes of such a model T are of equal length and sufficiently small, then the Cheeger constant of T is greater than 1, so the first nonzero Neumann eigenvalue μ_1 for T is greater than $\frac{1}{4}$. In the discussion of this in [4], T is assumed to have holes of equal length only to facilitate the construction of more complex surfaces by gluing along holes, and the proof applies without modification to the case of holes of unequal lengths, provided

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that the maximum length of a hole is sufficiently small. In more detail, the proof is obtained through a case examination of the ways in which a separating submanifold L can divide T, and the only cases in which the lengths of the holes of T enter into consideration occur if L either contains an arc connecting two points of different holes, or an arc connecting two points of the same hole which passes through the saddle between the other holes. In either case, the minimum length of such an arc tends to infinity as the maximum hole size tends to zero, and since the area of T is always 2π by the Gauss-Bonnet theorem, the bound follows for these cases. By our previous remarks, it follows that if λ_{2g-3} is sufficiently small, μ_1 must be greater than $\frac{1}{4}$.

The result for small $\lambda_1, \ldots, \lambda_{2g-3}$ now follows by estimating λ_{2g-2} from below by the corresponding element in the aggregate sequence composed of all Neumann eigenvalues of the three-holed spheres occurring in the decomposition of M, and noting that the zero eigenvalue is present in this sequence with multiplicity $2g-2$.

Remark. It is not difficult to see that μ_1 of a 3-holed sphere can be arbitrarily small provided the maximum hole size is not required to be small, which precludes the unrestricted use of the above approach in an attempt to determine, for example, whether or not λ_{2g-2} is always greater than $\frac{1}{4}$. This last question is perhaps a suitable subject for computer analysis. Since there are genus-dependent restrictions on the maximum hole size in a decomposition which minimizes this size [1, 5], an analysis of lower bounds for λ_{2g-2} along the above lines may be delicate. In this context I have learned that Buser, and independently Doyle, McMullen, and Thurston, have been able to show that there is a (currently) very small positive constant b such that $\lambda_{2g-2} > b$ for all $g \geq 2$.

To produce a 3-holed sphere with small μ_1, take a rectilinear geodesic pentagon P with one side very small. It is then straightforward to show that the first eigenvalue is close to zero for the mixed problem on P consisting of Dirichlet data on the very small side and Neumann data on the remaining four sides. Let φ be an eigenfunction for this eigenvalue. Reflect P through its short side to produce a rectilinear geodesic hexagon H, and note that the function $\tilde{\varphi}$ on H, which is the odd extension of φ to H through P's short geodesic, has mean zero on H and Neumann data on all sides of H. Now identify alternate sides of H to produce a 3-holed sphere S, and extend $\tilde{\varphi}$ to a function Φ on S by taking the even extension of $\tilde{\varphi}$ through the identified geodesics of H. Φ is then an eigenfunction for the Neumann problem on S having mean zero and an eigenvalue close to zero.

REFERENCES

1. Lipman Bers, *An inequality for Riemann surfaces*, Differential Geometry and Complex Analysis, Springer-Verlag, 1985.
2. Marc Burger, *Asymptotics of small eigenvalues of Riemann surfaces*, Bull. Amer. Math. Soc. **18** (1988), 39–40.
3. Peter Buser, *Riemannsche Flächen mit Eigenwerten in* $(0, \frac{1}{4}]$, Comment. Math. Helv. **52** (1977), 25–34.
4. ———, *On Cheeger's inequality* $\lambda_1 \geq h^2/4$, Geometry of the Laplace Operator, **36**, Proceedings of Symposia in Pure Mathematics, Amer. Math. Soc., Providence, RI, pp. 29–77.
5. ———, *Riemannsche Flächen und Längenspektrum vom trigonometrischen Standpunkt aus* (*Habilitationsschrift der Universität Bonn*), Universität Bonn.
6. Bruno Colbois, *Petites valuers propres du laplacien sur une surface de Riemann compacte et graphes.*, C. R. Acad. Sc. Paris **301** (1985), 927–930.
7. J. Dodziuk, T. Pignataro, B. Randol and D. Sullivan, *Estimating small eigenvalues of Riemann surfaces*, Contemp. Math. **64**, Amer. Math. Soc., Providence, RI, 1987, pp. 93–121.
8. J. Dodziuk and B. Randol, *Lower bounds for* λ_1 *on a finite-volume hyperbolic manifold*, J. Differential Geom. **24** (1986), 133–139.
9. T. Pignataro and D. Sullivan, *Ground state and lowest eigenvalue of the Laplacian for non-compact hyperbolic surfaces*, Comm. Math. Physics **104** (1986), 529–535.
10. Burton Randol, *Small eigenvalues of the Laplace operator on compact Riemann surfaces*, Bull. Amer. Math. Soc. **80** (1974), 998–1001.
11. R. Schoen, S. Wolpert and S.T. Yau, *Geometric bounds on the low eigenvalues of a compact surface.*, Geometry of the Laplace Operator, **36**, Proceedings of Symposia in Pure Mathematics, Amer. Math. Soc., Providence, RI, (1980), pp. 279–285.

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