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A SHORT PROOF OF ROHLIN'S THEOREM FOR COMPLEX SURFACES

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ABSTRACT. We prove that the signature of a two dimensional compact complex spin manifold is divisible by 16.

Rohlin [R] showed that the signature of a smooth closed oriented spin 4-manifold is divisible by 16. We give a proof for those manifolds which carry a complex structure. Recall that a manifold is spin if and only if the Stiefel-Witney class $w_2(X)$ vanishes. The key observation we need is that a complex manifold is spin if and only if its canonical bundle (the determinant of the holomorphic cotangent bundle) has a square root [A, 3.2]. By the dimension of a complex manifold, we mean its complex dimension.

Theorem. *If X is a two dimensional compact complex spin manifold then the signature $\sigma(X)$ is divisible by* 16.

Proof. Choose a holomorphic line bundle L such that $L \otimes L \cong K$, where K is the canonical bundle. Denote the holomorphic euler characteristic $\chi(O_X(L))$ by A. By Serre duality [S], the cup product gives a perfect pairing

$$H^i(X, O_X(L)) \times H^{2-i}(X, O_X(L)) \to H^2(X, O_X(K)) = \mathbf{C}.$$

Consequently $H^1(X, O_X(L))$ carries a nondegenerate skew-symmetric form, hence it is even dimensional. Therefore

$$A = 2\dim H^0(X, O_X(L)) - \dim H^1(X, O_X(L)) \equiv 0 \pmod 2.$$

On the other hand, by the Riemann-Roch and Hirzebruch signature theorems [AS],

$$\begin{aligned} A &= (c_1(L)^2 + c_1(L)\cdot c_1(X))/2 + (c_1(X)^2 + c_2(X))/12 \\ &= -(c_1(X)^2 - 2c_2(X))/24 \\ &= -\sigma(X)/8 \end{aligned}$$

which shows that $\sigma(X)$ has the required divisibility.

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Remark. The number A above is just the $\widehat{\mathbf{A}}$-genus; in general there is an identity

$$\widehat{\mathbf{A}}(X) = \exp(-c_1(X)/2)\,\mathrm{Todd}(X)$$

which shows that $\chi(O_X(L))$ equals the $\widehat{\mathbf{A}}$-genus for an arbitrary compact complex spin manifold. The above argument easily generalizes to show that $\chi(O_X(L))$, hence the $\widehat{\mathbf{A}}$-genus, is divisible by 2 for a complex spin manifold of dimension $4k+2$. This is a special case of [AH, Corollary 2].

References

[A] M. Atiyah, *Riemann surfaces and spin structures*, Ann. Sci. École. Norm Sup. **4** (1971), 47–62.

[AH] M. Atiyah and F. Hirzebruch, *Riemann-Roch theorem for differentiable manifolds*, Bull. Amer. Math. Soc. **65** (1959), 276–281.

[AS] M. Atiyah and I. Singer, *Index of elliptic operators* III, Ann. Math. **87** (1968), 546–604.

[R] V. A. Rohlin, *New results in the theory of* 4 *dimensional manifolds*, Dokl. Akad. Nauk. SSSR. **84** (1952), 221–224.

[S] J. P. Serre, *Un théoreme de dualite*, Com. Math. Helv. **29** (1955), 9–26.

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