A SHORT PROOF OF ROHLIN'S THEOREM FOR COMPLEX SURFACES

DONU ARAPURA

(Communicated by Frederick R. Cohen)

Abstract. We prove that the signature of a two dimensional compact complex spin manifold is divisible by 16.

Rohlin [R] showed that the signature of a smooth closed oriented spin 4-manifold is divisible by 16. We give a proof for those manifolds which carry a complex structure. Recall that a manifold is spin if and only if the Stiefel-Witney class \( w_2(X) \) vanishes. The key observation we need is that a complex manifold is spin if and only if its canonical bundle (the determinant of the holomorphic cotangent bundle) has a square root [A, 3.2]. By the dimension of a complex manifold, we mean its complex dimension.

Theorem. If \( X \) is a two dimensional compact complex spin manifold then the signature \( \sigma(X) \) is divisible by 16.

Proof. Choose a holomorphic line bundle \( L \) such that \( L \otimes L \cong K \), where \( K \) is the canonical bundle. Denote the holomorphic euler characteristic \( \chi(O_X(L)) \) by \( A \). By Serre duality [S], the cup product gives a perfect pairing

\[
H^i(X, O_X(L)) \times H^{2-i}(X, O_X(L)) \to H^2(X, O_X(K)) = \mathbb{C}.
\]

Consequently \( H^1(X, O_X(L)) \) carries a nondegenerate skew-symmetric form, hence it is even dimensional. Therefore

\[
A = 2 \dim H^0(X, O_X(L)) - \dim H^1(X, O_X(L)) \equiv 0 \pmod{2}.
\]

On the other hand, by the Riemann-Roch and Hirzebruch signature theorems [AS],

\[
A = \frac{(c_1(L)^2 + c_1(L) \cdot c_1(X))}{2} + \frac{(c_1(X)^2 + c_2(X))}{12} = -\frac{(c_1(X)^2 - 2c_2(X))}{24} = -\frac{\sigma(X)}{8}
\]

which shows that \( \sigma(X) \) has the required divisibility.

---

Received by the editors March 13, 1989.

1980 Mathematics Subject Classification (1985 Revision). Primary 57N13, 32J15.

This work was done at MSRI where the author was supported by NSF grant DMS-8505550.
Remark. The number $A$ above is just the $\tilde{A}$-genus; in general there is an identity

$$\tilde{A}(X) = \exp(-c_1(X)/2) \text{Todd}(X)$$

which shows that $\chi(O_X(L))$ equals the $\tilde{A}$-genus for an arbitrary compact complex spin manifold. The above argument easily generalizes to show that $\chi(O_X(L))$, hence the $\tilde{A}$-genus, is divisible by 2 for a complex spin manifold of dimension $4k + 2$. This is a special case of [AH, Corollary 2].

References


Department of Mathematics, Purdue University, West Lafayette, Indiana 47907