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ON THE STRUCTURE OF SEMIDERIVATIONS IN PRIME RINGS

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ABSTRACT. Let R be a prime ring. By a semiderivation associated with a function $g: R \to R$, we mean an additive mapping $f: R \to R$ such that, for all x, $y \in R$, $f(xy) = f(x)g(y) + xf(y) = f(x)y + g(x)f(y)$ and $f(g(x)) = g(f(x))$. It is known that g must necessarily be a ring endomorphism. Here it is shown that f must be an ordinary derivation or of the form $f(x) = \lambda(x - g(x))$ for all $x \in R$, where λ is an element of the extended centroid of R.

As a generalization of derivations, the following notion of semiderivations was introduced in Bergen [2]:

Definition. Let R be an associative ring. An additive mapping $f: R \to R$ is called a semiderivation associated with a function $g: R \to R$ if, for all x, $y \in R$,

(i) $f(xy) = f(x)g(y) + xf(y) = f(x)y + g(x)f(y)$;
(ii) $f(g(x)) = g(f(x))$.

Assume that R is a prime ring. If the semiderivation f does not vanish identically on R, it is shown in Chang [3] that the function g must necessarily be a ring endomorphism. Conversely, if g is a ring endomorphism of the prime ring R, then the mapping $f(x) = x - g(x)$ is a semiderivation of R associated with the ring endomorphism g. If the ring endomorphism g is the identity mapping, then all semiderivations associated with g are merely ordinary derivations. Our aim here is to show that any semiderivation of a prime ring R assumes essentially one of these two forms:

Theorem. *Let f be a semiderivation of a prime ring R associated with the (endomorphism) mapping $g: R \to R$. Then either one of the following two cases holds:*

(1) *There exists an element λ in the extended centroid of R such that $f(x) = \lambda(x - g(x))$ for all $x \in R$.*

(2) *The endomorphism g is an identity mapping and f is an ordinary derivation.*

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Our argument is essentially a refinement of Lemma 4 [1] and is crucially based on a clever computation of this lemma.

Proof. Set $f'(x) = x - g(x)$ for $x \in R$. Then f' is also a semiderivation of R associated with the ring endomorphism g. Let

$$U = \left\{ \sum_i u_i f(x_i) v_i \colon u_i, v_i, x_i \in R \text{ and } \sum_i u_i f'(x_i) v_i = 0 \right\}.$$

Then U is obviously a two-sided ideal of R. Let u_i, v_i, $x_i \in R$ be such that $\sum_i u_i f'(x_i) v_i = \sum_i u_i (x_i - g(x_i)) v_i = 0$. Applying the semiderivation f to $\sum_i u_i (x_i - g(x_i)) v_i = 0$ and using the defining identities (i), (ii) for the semiderivation f to expand the resulting expression, we compute, as in Lemma 4 [1]:

$$\begin{aligned}
0 &= f\left(\sum_i u_i f'(x_i) v_i\right) \\
&= f\left(\sum_i (u_i x_i v_i - u_i g(x_i) v_i)\right) \\
&= \sum_i [u_i f(x_i v_i) + f(u_i) g(x_i v_i) - f(u_i g(x_i)) g(v_i) - u_i g(x_i) f(v_i)] \\
&= \sum_i [u_i f(x_i) v_i + u_i g(x_i) f(v_i) + f(u_i) g(x_i) g(v_i) \\
&\qquad - f(u_i) g(x_i) g(v_i) - g(u_i) f(g(x_i)) g(v_i) - u_i g(x_i) f(v_i)] \\
&= \sum_i u_i f(x_i) v_i - \sum_i g(u_i) g(f(x_i)) g(v_i) \\
&= \sum_i u_i f(x_i) v_i - g\left(\sum_i u_i f(x_i) v_i\right).
\end{aligned}$$

Therefore

$$\sum_i u_i f(x_i) v_i = g\left(\sum_i u_i f(x_i) v_i\right)$$

whenever $\sum_i u_i f'(x_i) v_i = 0$. That is, $f'(u) = u - g(u) = 0$ for all $u \in U$. If the two-sided ideal U is nonzero, then by Lemma 1 [1], $f' = 0$ on R and hence $g(x) = x$ for $x \in R$. Thus g is the identity endomorphism of R and f is merely an ordinary derivation of R, as desired. Now, assume that $U = 0$. That is, for any u_i, v_i, $x_i \in R$, $\sum_i u_i f'(x_i) v_i = 0$ implies $\sum_i u_i f(x_i) v_i = 0$. Let W be the two-sided ideal

$$\left\{ \sum_i u_i f'(x_i) v_i \colon u_i, v_i, x_i \in R \right\}.$$

Then the mapping ϕ defined on W according to the rule

$$\phi \colon \sum_i u_i f'(x_i) v_i \to \sum_i u_i f(x_i) v_i,$$

where u_i, v_i, $x_i \in R$, is well defined. But ϕ is obviously an (R, R)-bimodule map of W into R. By the definition of the extended centroid of R, there exists an element λ in the extended centroid of R such that $\phi(w) = \lambda w$ for all $w \in W$. In particular, for u, v, $x \in R$, $uf(x)v = \phi(uf'(x)v) = \lambda(uf'(x)v) = u(\lambda f'(x))v$. It follows from the primeness of R that $f(x) = \lambda f'(x) = \lambda(x - g(x))$ for all $x \in R$, as asserted.

References

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