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THE C^*-ALGEBRAS OF REEB FOLIATIONS ARE NOT AF-EMBEDDABLE

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ABSTRACT. It is shown that the C^*-algebras of Reeb foliations cannot be embedded into AF-algebras.

A characterization of the C^*-algebra of the Reeb foliation of the solid torus is given in [W3, Example 9], which follows from [W3, Theorem 7]. The proof of that theorem uses groupoid equivalence. Here we give a more constructive proof. By using Pimsner's criterion [P], the results (Theorems 3, 5, 8, 9, 10) show the C^*-algebras of Reeb foliations can *not* be embedded into AF-algebras, in contrast to the case of the irrational rotation C^*-algebras [P-V]. Notice that every C^*-algebra of Reeb foliation is an extension, where both the ideal and the quotient can be embedded into AF-algebras! We recall that according to L. Brown, extensions of AF-algebras by AF-algebras are still AF-algebras [B]. Our examples show that in a certain sense, Brown's well-known result is the strongest possible.

The K-theory of C^*-algebras of Reeb foliations have already been studied by A. M. Torpe in [Tor]. Our characterization of the C^*-algebras can also be used to obtain alternative simpler proofs of the results of [Tor].

We shall consider C^*-algebras of foliated manifolds *with* boundaries, which are again defined as the C^*-algebras of corresponding holonomy groupoids (see [W2, §4.1]). We shall denote the annular, torus, solid torus, and the three sphere by R, T, ST, and S^3. Let $\mathscr{F}_R$ be the Reeb foliation of the corresponding manifold. Let $(R_0, \mathscr{F}_R)$ be the "half" of $(R, \mathscr{F}_R)$, with one side $\overline{O_1O_2}$ open (Figure 1). The Reeb foliation $(R, \mathscr{F}_R)$ admits a "transversal foliation," i.e., a foliation $(R, \mathscr{F}_T)$ whose leaves are transversals to $(R, \mathscr{F}_R)$ (Figure 1).

Let X' be a faithful transversal of $(R_0, \mathscr{F}_R)$ consisting of such a leaf (Figure 1). Let P be the intersection of X' with the closed leaf of $(R_0, \mathscr{F}_R)$. Let $X = X' \backslash \{P\}$.

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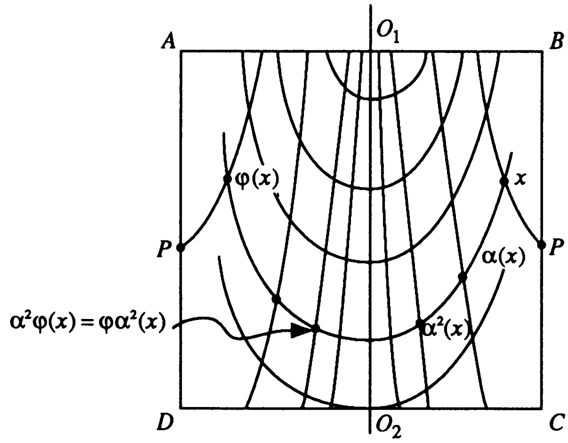

FIGURE 1.

Lemma 1. *There is a natural exact sequence*

$$0 \to C^*(G_x^x) \to C^*(G_{x'}^{x'}) \to C(S^1) \to 0. \tag{1}$$

Proof. The holonomy groups are trivial everywhere on X' except at p where $G_p^p \simeq \mathbf{Z}$; and $C^*(G_p^p) \simeq C^*(\mathbf{Z}) \simeq C(S^1)$. The exact sequence (1) follows from [W1, Proposition 1.2]. Q.E.D.

It is clear that $C^*(G_x^x) \simeq C(S^1) \otimes \mathscr{K}$. The structure of extension (1) is given by the following lemma. Notice that $C_0(0, \infty) \rtimes_\alpha \mathbf{Z} \simeq C(S^1) \otimes \mathscr{K}$ and the C^*-algebras are independent of a particular such α up to isomorphism.

Lemma 2. *There is a natural isomorphism*

$$\phi: C^*(G_x^x) \overset{\simeq}{\to} C_0[0, \infty) \rtimes_\alpha \mathbf{Z},$$

where $\alpha \in \mathrm{Diff}([0, \infty))$ *is a (and any) dilation with* 0 *as the only fixed point.*

Proof. Let α be the holonomy transformation on X. We define a map Φ: $C_c(G_x^x) \to C_c(\mathbf{Z}, C_c[0, \infty))$ by $\Phi f = (f_n)_{n \in \mathbf{Z}}$, where $f_n(x) = f(x, \alpha^n(x))$, for $x \in \mathbf{R}$. For $f, g \in C_c(G_x^x)$ one checks

$$\begin{aligned}(f * g)(x, \alpha^n(x)) &= \sum_{m \in \mathbf{Z}} f(x, \alpha^m(x)) g(\alpha^m(x), \alpha^n(x)) \\ &= \sum_{m \in \mathbf{Z}} f_m(x) \cdot g_{n-m}(\alpha^m(x)).\end{aligned}$$

On the other hand, for $\Phi f, \Phi g \in C_c(\mathbf{Z}, C_c[0, \infty))$,

$$(\Phi f * \Phi g)_n(x) = \sum_{m \in \mathbf{Z}} f_m(x) g_{n-m}(\alpha^m(x));$$

it is clear that Φ is an *-algebra homomorphism. It is also easy to show that Φ is actually an isometry. Thus Φ can be extended to the C^*-completions. Q.E.D.

By [H-S], it follows from Lemma 2 that

Theorem 3. *Let $(N_0, \mathscr{F})$ be a "half" of the annular with the Reeb foliation with one side open (as the region bounded by $\overline{BC}$ and $\overline{O_1O_2}$, with $\overline{O_1O_2}$ exclusive, Figure 1). Then*

$$C^*(N_0, \mathscr{F}) \simeq (C_0[0, \infty) \rtimes_\alpha \mathbf{Z}) \otimes \mathscr{K},$$

where $\alpha \in \mathrm{Diff}\,[0, \infty)$ is a dilation with 0 as the unique fixed point.

Let Y be the transversal of $(R, \mathscr{F}_R)$ consisting of two leaves of $(R, \mathscr{F}_T)$ on both sides (Figure 1). Identify one of them with $[0, \infty)$; then the points of the other leaf have coordinates $\phi(t)$, $t \in [0, \infty)$, where ϕ is the unique holonomy map given by $(R, \mathscr{F}_T)$ taking the first leaf onto the second.

Lemma 4.

$$C^*(G_y^y) \cong \left\{ f \in C_0([0, \infty), M_2(\mathbf{C})) \mid f(0) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\} \rtimes_\alpha \mathbf{Z},$$

where $\alpha \in \mathrm{Diff}\,[0, \infty)$ is a dilation with 0 as the unique fixed point.

Proof. Fix the holonomy map ϕ as above. We may choose the holonomy map α on the two sides in such a way that α on one side is the conjugation of α on the other by ϕ. Thus the two actions of $\mathbf{Z}$ and $\mathbf{Z}_2$ by α and ϕ commute. The holonomy groupoid

$$G_y^y = \left\{ (\alpha^m \phi^i(x), \alpha^n \phi^j(x)) \mid m, n \in \mathbf{Z},\ i, j \in \mathbf{Z}_2,\ x \in \mathbf{R} \right\}.$$

Let Φ be the linear map from $C_c(G_y^y)$ into $C_c(\mathbf{Z}, C_c([0, \infty), M_2(\mathbf{C})))$ such that

$$\Phi f_n(x) = \begin{pmatrix} f(x, \alpha^n(x)) & f(x, \phi\alpha^n(x)) \\ f(\phi(x), \alpha^n(x)) & f(\phi(x), \phi\alpha^n(x)) \end{pmatrix}, \qquad x \in [0, \infty),$$

for $n \in \mathbf{Z}$. If $f(x, \alpha^n(x)) = 0$ and $f(\phi(x), \alpha^n(x)) = 0$ for all x and n, then $f = 0$. So Φ is injective.

For any $f, g \in C_c(G_y^y)$, we have

$$\begin{aligned} f * g(\alpha^m \phi^i(x), \alpha^n \phi^j(x)) = \sum_{l \in \mathbf{Z}} [& f(\alpha^m \phi^i(x), \alpha^l(x)) g(\alpha^l(x), \alpha^n \phi^j(x)) \\ & + f(\alpha^m \phi^i(x), \alpha^l \phi(x)) g(\alpha^l \phi(x), \alpha^n \phi^j(x))] \end{aligned}$$

for $m, n \in \mathbf{Z}$, $i, j \in \mathbf{Z}_2$ and $x \in [0, \infty)$. A straightforward calculation yields

$$\begin{aligned}(\Phi f * \Phi g)_n(x) &= \int_{m\in\mathbf{Z}} \Phi f_m \alpha_m (\Phi g_{n-m})(x) \\ &= \sum_{m\in\mathbf{Z}} \begin{pmatrix} f(x, \alpha^m(x)) & f(x, \phi\alpha^m(x)) \\ f(\phi(x), \alpha^m(x)) & f(\phi(x), \phi\alpha^m(x)) \end{pmatrix} \\ &\qquad\qquad \times \begin{pmatrix} g(\alpha^m(x), \alpha^n(x)) & g(\alpha^m(x), \phi\alpha^n(x)) \\ g(\phi\alpha^m(x), \alpha^n(x)) & g(\phi\alpha^m(x), \phi\alpha^n(x)) \end{pmatrix} \\ &= \Phi(f * g)_n(x).\end{aligned}$$

Therefore Φ is an homomorphism. One checks again that Φ is actually isometric with respect to the C^*-norms after unwinding the definitions. We omit the routine arguments. Q.E.D.

Let $\mathbf{Z}$ act on $\mathscr{K}$ trivially and we get

Theorem 5.

$$C^*(N, \mathscr{F}_R) \simeq \left\{ f \in C_0([0, \infty), M_2(\mathscr{K})) | f(0) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\} \rtimes_\alpha \mathbf{Z},$$

where α *is given as in Lemma* 4.

We denote $\bar{\alpha}$ as the action ($\alpha\times$ trivial) on $[0, \infty) \times \mathbf{R}/\mathbf{Z}$.

The following lemma says that C^*-algebras of foliations are invariant under not too destructive surgery of the foliated manifold.

Lemma 6. *Let* $(M, \mathscr{F})$ *be a foliation, with dimension of leaves* $p > 2$. *Suppose* K *is a closed submanifold of* M *such that* $L_\alpha \cap (M\backslash K)$ *is still a* p*-dimensional submanifold of* $M\backslash K$ *for each leaf* L_α *of* $(M, \mathscr{F})$. *Then* $C^*(M\backslash K, \mathscr{F}) \simeq C^*(M, \mathscr{F})$.

Proof. There is a faithful transversal N of $(M\backslash K, \mathscr{F})$, that is, a transversal meeting every leaf of $(M\backslash K, \mathscr{F})$, with its dimension equal to the codimension of $(M\backslash K, \mathscr{F})$. Clearly N is also a faithful transversal of $(M, \mathscr{F})$ and [H-S] applies. Q.E.D.

An amazing consequence is

Lemma 7. *Let* $(C, \mathscr{F}_R)$ *be the infinite solid cylinder with Reeb foliation. Then*

$$C^*(C, \mathscr{F}_R) \simeq C_0[0, \infty) \otimes \mathscr{K}.$$

Proof. The infinite solid cylinder with Reeb foliation is conjugate to an infinite stack of open discs piled upon the origin in the plane whose origin is removed. Now apply Lemma 6. Q.E.D.

Theorem 8. *Let* $(ST, \mathscr{F}_R)$ *be the solid torus with Reeb foliation. Then*

$$C^*(ST, \mathscr{F}_R) \simeq C_0([0, \infty), \mathscr{K}) \rtimes_\alpha \mathbf{Z} = A_v,$$

where α *is as given in Theorem* 3.

Theorem 9. *Let* $(S^3, \mathcal{F}_R)$ *be the Reeb foliation. Then*

$$C^*(S^3, \mathcal{F}_R) \simeq A_v \vee A_v = \{(f_1, f_2) \in [C_0([0,\infty), \mathcal{K}) \rtimes_\alpha \mathbf{Z}]^2 | \pi(f_1) = \pi(f_2)\},$$

where α, π *are as given in Lemma* 1 *and Theorem* 3 *respectively.*

Let X be a compact metrizable space and T a homeomorphism of X. Using the techniques in [P-V], Pimsner gave a criterion for embedding the crossed product $C(X) \rtimes_T \mathbf{Z}$ into AF-algebras [P], that is, it is embeddable if and only if every point of X is "pseudo-non-wandering". Unfortunately, in our characterization of C^*-algebras of Reeb foliations, the topological spaces X are not compact, so Pimsner's criterion does not apply. However, by the same technique, we can show our next result.

Theorem 10. *C^*-algebras of Reeb foliations can not be embedded into AF-algebras.*

Proof. We need only to consider $C^*(R_0, \mathcal{F}_R)$, which is a subalgebra of the others. By Theorem 3, it is enough to show that $C_0[0,\infty) \rtimes_T \mathbf{Z}$ is not embeddable.

Let π_t be the regular representation of $C_0[0,\infty) \rtimes_T \mathbf{Z}$ on $H_t = l^2(\mathbf{Z})$ associated to the evaluation representation $\sigma_t \colon f \to f(t)$ of $C_0[0,\infty)$ for each $t \in [0,\infty)$. Then $\bigoplus_{t\in[0,\infty)} \pi_t$ is a faithful representation of $C_0[0,\infty) \rtimes_T \mathbf{Z}$ since $\bigoplus_{t\in[0,\infty)} \sigma_t$ is faithful for $C_0[0,\infty)$. We have

$$\pi_t(f)e_n = f(T^n t)e_n,$$
$$\pi_t(u)e_n = e_{n+1},$$

where e_n are the canonical basis of $l^2(\mathbf{Z})$, the funtion $f \in C_0(X)$ and u is the generator of $\mathbf{Z}$.

Define $f \in C_0[0,\infty)$ by letting $f = 2$ on $[0,1]$, $f(n) = 1/2^{n-1}$ for $n = 2, 3, 4, \ldots$, and interpolating linearly the value of $f(t)$ for nonintegers $t \geq 1$. We claim that for every $t \in [0,\infty)$, the operator $I - \pi_t(fu)$ is injective on the Hilbert space H_t. Suppose $v = \sum_{-\infty}^{\infty} a_n e_n$, and that $(1 - \pi_t(fu))v = 0$. Then $a_{n+1} = a_n f(T^{n+1}t)$, and for all n large enough $f(T^{n+1}t) = 2$, thus $v \notin l^2(\mathbf{Z})$. Thus the operator $S = \bigoplus_{t\in[0,\infty)}(1 - \pi_t(fu))$ is injective on $\bigoplus_{t\in[0,\infty)} H_t$. Notice the transpose

$$(I - \pi_t(f \cdot u))^* e_n = e_n - f(T^n t)e_{n-1},$$

and we can choose $v_t = \sum a_n e_n \in H_t$ such that $a_0 = 1$, while $a_n = f(T^n t)^{-1} \cdot a_{n-1}$ for $n \neq 0$. Therefore $v_t \in \ker(I - \pi_t(f \cdot u))^*$, for $t \in (0,\infty)$. Thus S is not surjective. It is easy to see that the polar decomposition of S yields a nonunitary isometry in $\bigoplus_{t\in[0,\infty)} \pi_t(C_0[0,\infty) \rtimes \mathbf{Z})$.

By P. Halmos' theorem (cf. [P, Theorem 9]), the crossed product $C_0[0,\infty) \rtimes_\alpha \mathbf{Z}$ is not quasidiagonal and therefore not embeddable into AF-algebras, because the latter are always quasidiagonal. Q.E.D.

Assume that $\mathbf{Z}$ acts on itself by translation. Then $C_0(\mathbf{Z}) \rtimes \mathbf{Z} \simeq \mathscr{K}$ is of course an AF-algebra. By [P] if X is the one point compactification of $\mathbf{Z}$, then $C(X) \rtimes \mathbf{Z}$ is embeddable. Here we saw that if X is only one-side-compactified, $C_0(X) \rtimes \mathbf{Z}$ is not embeddable.

Added in proof. For more information about embedding C^*-algebra extensions into AF-algebras, the reader is referred to J. S. Spielberg's paper [S].

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