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INVARIANT SUBSPACE OF STRICTLY SINGULAR OPERATORS

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ABSTRACT. In this paper, we show that strictly singular operators are condensing maps. Moreover, we obtain a new result that every bounded linear operator T on a Banach space that commutes with a nonzero strictly singular operator S has a non-trivial invariant closed subspace.

1. INTRODUCTION

The famous problem that has challenged mathematicians for many years is whether every bounded linear operator on a Banach space has a non-trivial closed invariant subspace. Except for a few special classes of operators (such as the self-adjoint operators on a Hilbert space), it was unsolved for general Banach spaces until 1954, when N. Aronszajn and K. Smith proved that every compact operator has a non-trivial closed subspace. Twelve years later, A. Bernstein and A. Robinson, using the method of nonstandard analysis, extended this result to an operator T such that $p(T)$ is compact for some nonzero polynomial $p(t)$. Still later on, in 1973, V. I. Lomonosov published a remarkable generalization of the Bernstein-Robinson Theorem (see [2]). In this paper, we use the method of non-linear functional analysis to extend this result to an operator T commuting with a strictly singular operator S.

We denote the set of all linear bounded operators from a normed space X to a normed space Y by $B(X, Y)$, the set of all strictly singular operators by $S(X, X)$ (see [4]), the set of all elements of $B(X, X)$ commuting with an operator B by $\mathfrak{U}_B$, the convex hull of a set A by $\operatorname{co}(A)$, and the closure of A by $\overline{A}$. $x \in A - B$ means $x \in A$ and $x \notin B$.

Let us begin with some concepts and properties that we shall meet in this paper. Most of them are well known and a few unfamiliar ones are referred to [3, §§16.1 and 16.2].

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Each metric space X is paracompact and for each open covering of X there is a partition of unity subordinated to it.

Let A be a bounded subset of a metric space X. We put

$$\gamma(A) = \inf\{d\colon \text{There is a finite number of subsets } S_i \text{ such that } \bigcup_{i=1}^{n} S_i \supset A, \text{ and } \operatorname{diam}(S_i) \le d,\ i = 1, \dots, n.\}.$$

$\gamma(A)$ is called the measure of noncompactness of A, and it has the following properties:

1. $\gamma(A) = 0$ iff $\overline{A}$ is a compact set;
2. for each scalar t, $\gamma(tA) = |t|\gamma(A)$; $\gamma(\overline{A}) = \gamma(A)$, $\gamma(\operatorname{co}(A)) = \gamma(A)$;
3. for arbitrary bounded subsets A, B of X,

$\gamma(A+B) \le \gamma(A)+\gamma(B)$, $\gamma(A\cup B) = \max(\gamma(A), \gamma(B))$, and $\gamma(A) \le \gamma(B)$ if $A \subset B$.

Further, let X, Y be metric spaces, and $F\colon X \to Y$ be a bounded continuous map. We call F a condensing map if $\gamma(F(A)) < \gamma(A)$ for each bounded subset of X with $\gamma(A) \neq 0$. We call a bounded linear operator $T\colon X \to Y$ strictly singular if the restriction of T to any infinite-dimensional subspace of X is not an isomorphism (see [4]).

According to [3, Theorem 16.12], we have the following fact:
Let X be a Banach space and A an open subset of X. Let $F\colon \overline{A} \to X$ be a continuous bounded map satisfying the following conditions:

(i) There is $x_0 \in A$ such that $F(x) - x_0 \neq \alpha(x - x_0)$ for each $x \in \partial A$ and $\alpha > 1$;

(ii) F is a condensing map.

Then F has at least one fixed point $\bar{x} \in \overline{A}$.

2. Main result

Lemma 1. *Let B be the balanced hull of A; then $\gamma(B) = \gamma(A)$.*

Proof. Since $B = \bigcup_{|z|\le 1}(zA)$, for each $\varepsilon > 0$, we can find $z_1, \dots, z_n$ such that, for $|z| \le 1$, there is z_i satisfying $|z - z_i| < \varepsilon$. Put

$$D_i = \{z\colon |z| \le 1,\ |z - z_i| < \varepsilon\}.$$

Then

$$\gamma\left(\bigcup_{z\in D_i}(zA)\right) \le \gamma(z_i A) + \varepsilon\gamma(A) \le \gamma(A) + \varepsilon\gamma(A),$$

and $B = \bigcup_{i=1}^{n}(\bigcup_{z\in D_i}(zA))$. So $\gamma(B) = \max\{\gamma(\bigcup_{z\in D_i}(zA)),\ i = 1, \dots, n\} \le \gamma(A) + \varepsilon\gamma(A)$, that is, $\gamma(B) \le \gamma(A)$. Clearly, $\gamma(A) \le \gamma(B)$; thus we have $\gamma(B) = \gamma(A)$.

Lemma 2. *Let X, Y be Banach spaces and $T \in B(X, Y)$. The following statements are equivalent*:

(i) *T is a one to one map and there is a constant $C > 0$ such that $\gamma(T(D)) \ge C\gamma(D)$ for each bounded subset D of X.*

(ii) *$T\colon X \to Y$ is an isomorphism.*

Proof. (i) $\Rightarrow$ (ii). Assume that T is not an isomorphism. There is a sequence in S_x, the closed unit sphere in X, such that $\lim \|Tx_n\| = 0$. We see that $\overline{\{x_n\}}_{n=1}^{\infty}$ is not a compact set. In fact, if there is $x_0 \in S_x$ such that $x_{n_k} \to x_0$, we would have $\|Tx_0\| = \lim \|Tx_{n_k}\| = 0$, which is contrary to the assumption. Therefore $\gamma(\{x_n\}_{n=1}^{\infty}) \neq 0$. However, $\overline{\{Tx_n\}}_{n=1}^{\infty}$ is a compact set and so $\gamma(\{Tx_n\}_{n=1}^{\infty}) = 0$. This again is contrary to the assumption.

(ii) $\Rightarrow$ (i). Let D be a bounded subset of X with $\gamma(D) \neq 0$. For any finite number of subsets S_i of Y with $\bigcup_i S_i \supset TD$ and $\operatorname{diam}(S_i) \le d$ for all i, we have $\bigcup_i T^{-1}(S_i) \supset D$, and $\operatorname{diam}(T^{-1}(S_i)) \le \|T^{-1}\| d$, and hence $d \ge \|T^{-1}\|^{-1}\gamma(D)$. By definition we have $\gamma(T(D)) \ge \|T^{-1}\|^{-1}\gamma(D)$. For $\gamma(D) = 0$, (i) follows trivially from (ii).

Lemma 3. *Let X be a normed space and $T \in B(X, X)$. Let D be a subset of X satisfying the conditions:*

(i) *For each pair $x, y \in D$, $x \neq y$ implies $Tx \neq Ty$;*

(ii) $\gamma(T(D)) \ge \gamma(D)$.

Then $\gamma(D_1) \le \gamma(T(D_1))$ for each subset D_1 of D.

Proof. The conclusion is trivial if $\gamma(D) = 0$. We prove it for $\gamma(D) \neq 0$. For each $x \in D - D_1$ with $D_1 \subset D$ properly, we have $D = (D - \{x\}) \cup \{x\}$ and $TD = T(D - \{x\}) \cup \{Tx\}$.

Further, $\gamma(D-\{x\}) \le \gamma(T(D-\{x\}))$ and $\gamma(D-\{x\}) \neq 0$, since $\gamma(D-\{x\}) = \gamma(D)$. We put

$$W = \{V : D_1 \subset V \subset D, \gamma(T(V)) \ge \gamma(V)\}.$$

Clearly, W is nonempty. We define a partial order relation on W as follows: For $V_1, V_2 \in W$, $V_1 < V_2$ iff $V_1 \supset V_2$ properly.

For any totally ordered nonempty subset M of W, put $V_0 = \bigcap_{v \in M} V$. We assert that V_0 is an upper bound of M. Actually, we have $\inf_{v \in M} \gamma(TV) \ge \gamma(TV_0)$, $\inf_{v \in M} \gamma(V) \ge \gamma(V_0)$, and $\inf_{v \in M} \gamma(TV) \ge \inf_{v \in M} \gamma(V)$. We need only to show $\gamma(TV_0) = \inf_{v \in M} \gamma(TV)$. In fact, if otherwise, we let $\varepsilon_0 = \inf_{v \in M} \gamma(TV) - \gamma(TV_0) > 0$. Then we have $\gamma(T(V - V_0)) \ge \gamma(TV_0) + \varepsilon_0 \ge \varepsilon_0$ for all $V \in M$. Since $M \neq \varnothing$ and is totally ordered, so is the set $\{T(V - V_0) : V \in M\}$. It follows that $\gamma(\bigcap_{v \in M}(T(V - V_0))) \neq 0$ since $\gamma(T(V - V_0)) \ge \varepsilon_0$ for all $V \in M$. By the assumption (i), $\bigcap_{v \in M}(T(V - V_0)) = T(\bigcap_{v \in M}(V - V_0)) = \varnothing$, which is impossible; i.e. $\gamma(TV_0) = \inf_{v \in M} \gamma(TV)$. It follows that $\gamma(TV_0) \ge \gamma(V_0)$. So V_0 is an upper bound of M.

By Zorn's Lemma, there exists a maximal element D_0 in W. We see that $D_0 = D_1$. Otherwise, there would be an $x \in D_0 - D_1$, and $D_0 - \{x\} \in W$. This is contrary to the maximality of D_0 in W.

Lemma 4. *Let X, Y be Banach spaces and $T \in B(X, Y)$. If $\gamma(A) \le \gamma(TA)$ holds for each bounded subset A of X, then* $\operatorname{diam} T^{-1}(0) < +\infty$.

Proof. Assume that $\operatorname{diam} T^{-1}(0) = +\infty$. Then the closed unit sphere in $T^{-1}(0)$ is not totally bounded, and we can find an infinite sequence $\{x_n\}_{n=1}^{\infty}$ in this sphere contained in $T^{-1}(0)$ which has no Cauchy subsequence, and hence $\overline{\{x_n\}_{n=1}^{\infty}}$ is not compact. Then $\gamma(\{x_n\}_{n=1}^{\infty}) \neq 0$, but $\gamma(\{Tx_n\}_{n=1}^{\infty}) = 0$. This is contrary to the assumption.

Theorem 1. *Let X be a Banach space and $T \in B(X, X)$. Then $T \in S(X, X)$ implies that T is a condensing map.*

Proof. Suppose T is not condensing; there would be a bounded subset D of X with $\gamma(D) \neq 0$, such that $\gamma(TD) \geq \gamma(D)$. We may choose a subset of D, say D_1, such that $\gamma(D_1) \neq 0$, $\gamma(TD_1) \geq \gamma(D_1)$ and $Tx \neq Ty$ for all x, $y \in D_1$ when $x \neq y$.

Let A be the absolutely convex hull of $D_1 \cup \{0\}$; then A satisfies the conditions (i) and (ii) of Lemma 3. Since $\overline{A}$ is an absolutely convex neighborhood in $Z = \overline{\operatorname{span}(A)}$, then for each bounded subset G of Z, there exists a constant $\lambda > 0$ such that $\lambda\overline{A} \supset G$. It follows that $\gamma(TG) \geq \gamma(G)$ from Lemma 3 and $\gamma(T(\lambda\overline{A})) \geq \gamma(\lambda\overline{A})$.

We put $T_1 = T|_Z$, the restriction of T to Z. Since G is arbitrary, we see that $\dim T_1^{-1}(0) < +\infty$ by Lemma 4. There is a closed subspace Z_1 of Z, that is complementary to $T_1^{-1}(0)$, i.e. $Z = T_1^{-1}(0) \oplus Z_1$. Thus $T|_{Z_1} = T_1|_{Z_1}$ is injective. Therefore, $T|_{Z_1}$ is an isomorphism by Lemma 2. This is contrary to the fact that T is a strictly singular operator.

Theorem 2. *Let T belong to $B(X, X)$ and commute with a nonzero element S of $S(X, X)$. If $\{\mathfrak{U}_T x\}$ is dense in X for each nonzero element x of X, there exists an element T_0 of $\mathfrak{U}_T$ such that 1 is an eigenvalue of $T_0 S$.*

Proof. Since $S \neq 0$, we may choose $x_0 \in X$ with $\delta = \|Sx_0\| - \|S\| > 0$. Let $G = \{x: \|x - x_0\| < 1\}$, and $\overline{G} = \{x: \|x - x_0\| < 1\}$. We have then $\|Sx\| > 0$ for all $x \in \overline{G}$. By the assumption $\{\mathfrak{U}_T y\}$ is dense in X for all $y \in S\overline{G}$. Hence there is a $T_y \in \mathfrak{U}_T$ such that $\|T_y y - x_0\| < 1$. Let $\varepsilon_y = (1 - \|T_y y - x_0\|)/\|T_y\| > 0$. Then $\|T_y z - x_0\| < 1$ for all z in $O(y, \varepsilon_y)$, where $O(y, \varepsilon_y) = \{z: \|y - z\| < \varepsilon_y\}$.

The family $\{O(y, \varepsilon_y): y \in S\overline{G}\}$ is an open covering of $\overline{S\overline{G}}$. Let $\{f_y: y \in S\overline{G}\}$ be a partition of unity subordinated to $\{O(y, \varepsilon_y): y \in S\overline{G}\}$, and let

$$\varphi(z) = \sum_{y \in S\overline{G}} f_y(z) T_y z \qquad \text{for all } z \in S\overline{G}.$$

Then φ is a bounded continuous map from $S\overline{G}$ to $\overline{G}$. Again, let

$$\psi(x) = \sum_{y \in S\overline{G}} f_y(Sx) T_y Sx \qquad \text{for all } x \in \overline{G}.$$

Then ψ is a bounded continuous map from $\overline{G}$ to $\overline{G}$.

For each $x \in \partial G$, we have the following estimate:

$$\|\psi(x) - x_0\| = \left\| \sum_{y \in S\overline{G}} f_y(Sx) T_y Sx - x_0 \right\| = \left\| \sum{}' f_y(Sx)(T_y Sx - x_0) \right\| < 1,$$

where $\sum'$ means the sum of the terms in which Sx belongs to the support of f_y. We have $\psi(x) - x_0 \neq \alpha(x - x_0)$ for each $x \in \partial G$ and all $\alpha > 1$.

Moreover, for each bounded subset G_1 of G with $\gamma(G_1) \neq 0$, we show that $\gamma(\psi(G_1)) < \gamma(G_1)$. In fact, let us put $U_y = \overline{f_y(SG_1)}$. Then U_y is a compact set and the product $\prod_{y \in S\overline{G}} U_y$ is compact in $\prod_{y \in S\overline{G}} E_y$ by Tychonoff's Theorem, where all E_y are one-dimensional Euclidean spaces.

For each $x \in G_1$, $f_y(Sx) = 0$ for all but finitely many y, say $y_1, \dots, y_n$, such that $f_{y_i}(Sx) \neq 0$, $i = 1, \dots, n$. For arbitrary $\varepsilon > 0$, we put

$$U_i = \{r : r \in E_1, |f_{y_i}(Sx) - r| < \varepsilon/2^i\}.$$

Then $U(x) = U_1 \times U_2 \cdots \times U_n \times \prod_{y \in S\overline{G} - \{y_1, \dots, y_n\}} E_y$ is an open set and $\{U(x) : x \in G_1\}$ is an open covering of $\prod_{y \in S\overline{G}} U_y$. Therefore, we may choose $x_1, \dots, x_m$ such that $\{U(x_i) : i = 1, \dots, m\}$ is also an open covering. We put

$$G_{1i} = \{x : x \in G_1, \{f_y(Sx)\}_{y \in S\overline{G}} \in U(x_i)\}, \qquad i = 1, \dots, m.$$

Then $G_1 = G_{11} \cup G_{12} \cup \cdots \cup G_{1m}$. We assert that

$$\bigcup_{x \in G} \left\{ \sum_{y \in S\overline{G}} f_y(Sx)(T_y S)x \right\} \subset \sum_{y \in S\overline{G}} f_y(Sx_i)(T_y S)(G_{1i}) + C\varepsilon O(0, 1)$$

for some constant $C > 0$. In fact, for each $x \in G_{1i}$, we have $\sum_{j=1}^n |f_y(Sx) - f_{y_j}(Sx_i)| < \varepsilon$, hence

$$\sum_{y \in S\overline{G} - \{y_1, \dots, y_n\}} f_y(Sx) < \varepsilon$$

and

$$\sum_{y \in S\overline{G} - \{y_1, \dots, y_n\}} f_y(Sx) T_y Sx \in \varepsilon G.$$

This implies that

$$\sum_{y \in S\overline{G}} f_y(Sx) T_y Sx \in \sum_{y \in S\overline{G}} f_y(Sx_i) T_y Sx + \varepsilon O(0, 1) + \varepsilon G.$$

Furthermore,

$$\bigcup_{x \in G_{1i}} \left\{ \sum_{y \in S\overline{G}} f_y(Sx) T_y Sx \right\} \subset \sum_{y \in S\overline{G}} f_y(Sx_i)(T_y S)(G_{1i}) + C\varepsilon O(0, 1),$$

where $C = 2(1 + \|x_0\|)$.

From the preceding arguments, we obtain

$$\begin{aligned}\gamma(\psi(G_1)) &\leq \gamma\left(\bigcup_{j=1}^{m}\left(\sum_{y\in S\overline{G}} f_y(Sx_i)T_yS(G_{1j})\right) + C\varepsilon O(0,1)\right)\\ &\leq \max\left\{\gamma\left(\sum_{y\in S\overline{G}} f_y(Sx_i)T_yS(G_{1j})\right) + 2C\varepsilon\colon j=1,\ldots,m\right\}\\ &\leq \max\left\{\sum_{y\in S\overline{G}} f_y(Sx_i)\gamma(T_yS(G_{1j})) + 2C\varepsilon\colon j=1,\ldots,m\right\}\\ &< \max\left\{\sum_{y\in S\overline{G}} f_y(Sx_i)\gamma(G_{1j}) + 2C\varepsilon\colon j=1,\ldots,m\right\} \qquad \text{(Theorem 1)}\\ &= \max\{\gamma(G_{1j}) + 2C\varepsilon\colon j=1,\ldots,m\} \leq \gamma(G_1) + 2C\varepsilon.\end{aligned}$$

By the arbitrariness of ε, we have $\gamma(\psi(G_1)) < \gamma(G_1)$ when $\gamma(G_1) \neq 0$. We get a fixed point $\bar{x}$ of ψ in $\overline{G}$, that is $\psi(\bar{x}) = \bar{x}$. Let $T_0 = \sum_{y\in S\overline{G}} f_y(S\bar{x})T_y$; then $T_0 \in \mathfrak{U}_T$. Clearly, 1 is an eigenvalue of T_0S.

Theorem 3. *Let T belong to $B(X,X)$ and $T \neq \alpha I$. If T commutes with a nonzero strictly singular operator S, then there is a nontrivial closed subspace of X that is invariant under each operator in $\mathfrak{U}_T$.*

The proof of this theorem is completely similar to that of Lomonosov's Theorem and is omitted here.

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