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DAUGAVET'S EQUATION AND ORTHOMORPHISMS

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ABSTRACT. The main result of this paper asserts that every Dunford-Pettis operator on an AL-space having no discrete elements satisfies Daugavet's equation $\|I + T\| = 1 + \|T\|$; this extends a recent result of Holub on weakly compact operators. The proof is based on properties of orthomorphisms on a Banach lattice which also yield a short proof of another result of Holub concerning Daugavet's equation for bounded operators on an arbitrary AL- or AM-space.

1. INTRODUCTION

A linear operator $T: \mathbf{E} \to \mathbf{E}$ on a Banach space $\mathbf{E}$ satisfies *Daugavet's equation* if it satisfies

$$\|I + T\| = 1 + \|T\|,$$

where $I: \mathbf{E} \to \mathbf{E}$ denotes the identity operator. Daugavet's equation clearly fails for $T := -I$, but it holds under suitable conditions on $\mathbf{E}$ and T.

The first result on Daugavet's equation is due to Daugavet [6] who proved that the identity $\|I + T\| = 1 + \|T\|$ holds for every compact operator on $C[0,1]$. This result was subsequently extended into various directions [1, 4, 5, 7–13, 17, 18]; in particular, it follows from results of Foias and Singer [8] and Holub [9, 10] that Daugavet's equation holds for every weakly compact operator on $C[0,1]$ or $L^1[0,1]$, and that every bounded operator on these spaces satisfies $\max\{\|I+T\|, \|I-T\|\} = 1 + \|T\|$. It is remarkable that, with the exception of the results due to Foias and Singer [8] and Krasnoselskii [12], all known results on Daugavet's equation concern linear operators on a *Banach lattice*, whereas Banach lattice methods have only been used by Lozanovskii [13], Synnatzschke [17, 18], and Abramovich [1].

In the present paper we propose a unifying approach to the study of Daugavet's equation for linear operators on a Banach lattice. This approach is based on the properties of orthomorphisms which turn out to provide an efficient tool for extending results and unifying their proofs. Among other results, we shall

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prove that every bounded operator on an arbitrary AL- or AM-space satisfies $\max\{\|I+T\|, \|I-T\|\} = 1 + \|T\|$, and that Daugavet's equation holds for every Dunford-Pettis operator on an AL-space or a Dedekind complete AM-space with unit if and only if the Banach lattice contains no discrete elements.

Throughout this paper, let $\mathbf{E}$ be a Banach lattice, let $\mathscr{L}(\mathbf{E})$ denote the normed ordered vector space of all bounded operators $\mathbf{E} \to \mathbf{E}$, and let $I: \mathbf{E} \to \mathbf{E}$ denote the identity operator. A linear operator $Q: \mathbf{E} \to \mathbf{E}$ is an *orthomorphism* if it is order bounded and if $Q(B) \subseteq B$ holds for each band B of $\mathbf{E}$. Let $\mathrm{Orth}(\mathbf{E})$ denote the Riesz space [3, Theorem 8.9] of all orthomorphisms $\mathbf{E} \to \mathbf{E}$. If $\mathbf{E}$ is an AL-space or a Dedekind complete (= order complete [14]) AM-space with unit, then $\mathscr{L}(\mathbf{E})$ is a Dedekind complete Banach lattice [14, Theorem IV.1.5]; in particular, for each $T \in \mathscr{L}(\mathbf{E})$, the modulus $|T| := T \vee (-T)$ exists and satisfies $\||T|\| = \|T\|$. Moreover, under the same condition on $\mathbf{E}$, $\mathrm{Orth}(\mathbf{E})$ is a Dedekind complete AM-space with unit I and agrees with the band generated by I in $\mathscr{L}(\mathbf{E})$ [3, Theorems 15.5 and 8.11]. These properties of $\mathrm{Orth}(\mathbf{E})$ combined with Lemma 2.3 below indicate a natural connection between Daugavet's equation and orthomorphisms.

2. Bounded operators

We start with a simple but useful lemma on positive operators:

2.1. **Lemma.** *Let $\mathbf{E}$ be an AL- or AM-space. Then Daugavet's equation holds for every positive operator $\mathbf{E} \to \mathbf{E}$.*

Proof. Assume first that $\mathbf{E}$ is an AL-space and consider a positive operator $T: \mathbf{E} \to \mathbf{E}$. Then

$$\|(I+T)z\| = \|z\| + \|Tz\|$$

holds for each $z \in \mathbf{E}_+$, and this yields

$$\|I+T\| = 1 + \|T\|.$$

Assume now that $\mathbf{E}$ is an AM-space. Then $\mathbf{E}'$ is an AL-space, and the assertion follows by duality. □

Without the positivity assumption, we have the following result:

2.2. **Theorem.** *Let $\mathbf{E}$ be an AL- or AM-space. Then*

$$\max\{\|I+T\|, \|I-T\|\} = 1 + \|T\|$$

holds for each $T \in \mathscr{L}(\mathbf{E})$.

Proof. Let us first assume that $\mathbf{E}$ is a Dedekind AM-space with unit $e \in \mathbf{E}_+$. For each $U \in \mathrm{Orth}(\mathbf{E})$, we have $|I+U| \vee |I-U| = I + |U|$, and thus

$$\begin{aligned} \max\{\|I+U\|, \|I-U\|\} &= \||I+U| \vee |I-U|\| \\ &= \|I+|U|\| \\ &= 1 + \|U\|, \end{aligned} \tag{1}$$

by [3, Theorem 15.5] and Lemma 2.1. Consider now $T \in \mathscr{L}(\mathbf{E})$ and choose $S \in \mathrm{Orth}(\mathbf{E})$ and $R \in \mathrm{Orth}(\mathbf{E})^{\perp}$ satisfying

$$T = S + R$$

[3, Theorem 8.11]. Since $|R|e$ is dominated by a scalar multiple of e, there exists a positive $Q \in \mathrm{Orth}(\mathbf{E})$ satisfying

$$Qe = |R|e$$

[3, Theorem 8.15]. Moreover, for each $P \in \mathrm{Orth}(\mathbf{E})$, we have

$$|P + Q| \vee |P - Q| = |P| + Q$$

and

$$|P| + |R| = |P + R| = |P - R|,$$

hence

$$(|P + Q| \vee |P - Q|)e = |P + R|e = |P - R|e,$$

and thus

$$\max\{\|P + Q\|, \|P - Q\|\} = \|P + R\| = \|P - R\|. \tag{2}$$

Replacing P by S, $I + S$, and $I - S$ in (2), we obtain

$$\begin{aligned} \max\{\|S + Q\|, \|S - Q\|\} &= \|T\|, \\ \max\{\|I + S + Q\|, \|I + S - Q\|\} &= \|I + T\|, \\ \max\{\|I - S + Q\|, \|I - S - Q\|\} &= \|I - T\|; \end{aligned}$$

similarly, replacing U by $S + Q$ and $S - Q$ in (1), we obtain

$$\begin{aligned} \max\{\|I + S + Q\|, \|I - S - Q\|\} &= 1 + \|S + Q\|, \\ \max\{\|I + S - Q\|, \|I - S + Q\|\} &= 1 + \|S - Q\|. \end{aligned}$$

This yields

$$\begin{aligned} &\max\{\|I + T\|, \|I - T\|\} \\ &\qquad = \max\{\|I + S + Q\|, \|I + S - Q\|, \|I - S + Q\|, \|I - S - Q\|\} \\ &\qquad = 1 + \|T\|. \end{aligned}$$

In the case where $\mathbf{E}$ is an AL-space or an arbitrary AM-space, the assertion follows by duality. □

Theorem 2.2 is, up to an application of the representation theorems for AL- and AM-spaces, essentially due to Holub [9, 10]; see also Abramovich [1] for a short proof of Holub's result.

We conclude this section with another consequence of Lemma 2.1 which will be essential in what follows:

2.3. **Lemma.** *Let* $\mathbf{E}$ *be an AL-space or a Dedekind complete AM-space with unit. Then Daugavet's equation holds for each* $T \in \mathscr{L}(\mathbf{E})$ *satisfying* $I \wedge |T| = 0$.

Proof. By assumption, we have $|I + T| = I + |T|$, and thus

$$\|I + T\| = \|I + |T|\| = 1 + \|T\|,$$

by Lemma 2.1. □

Since Daugavet's equation holds for $T := I$, the condition of Lemma 2.3 is only sufficient but not necessary for Daugavet's equation to hold.

3. Dunford-Pettis operators

An element $u \in \mathbf{E}_+$ is *discrete* if the subspace and the ideal generated by $\{u\}$ in $\mathbf{E}$ agree.

If $\mathbf{E}$ is Dedekind complete, then a linear operator $\mathbf{E} \to \mathbf{E}$ is *AM-compact* if it maps the order bounded subsets of $\mathbf{E}$ into the relatively compact order bounded subsets of $\mathbf{E}$; see [19, p. 505].

3.1. **Lemma.** *If* $\mathbf{E}$ *is Dedekind complete and contains no discrete elements, then* $S = 0$ *holds for every positive AM-compact orthomorphism* $S\colon \mathbf{E} \to \mathbf{E}$.

Proof. Consider $z \in \mathbf{E}_+$. For each $y \in [0, Sz]$, there exists some $Q \in \operatorname{Orth}(\mathbf{E})$ satisfying $y = QSz$ and $0 \le Q \le I$, by [3, Theorem 8.15 and its proof]. Since S is an orthomorphism, we have

$$y = QSz = SQz \in S[0, z],$$

by [3, Theorems 8.24 and 8.21]. This yields

$$[0, Sz] = S[0, z].$$

Since S is AM-compact, this implies that the order interval $[0, Sz]$ is compact, hence Sz belongs to the band generated by the discrete elements in $\mathbf{E}$ [2, Theorem 21.12], and the assumption on $\mathbf{E}$ yields $Sz = 0$. □

A linear operator $\mathbf{E} \to \mathbf{E}$ is a *Dunford-Pettis* operator if it maps the weakly compact subsets of $\mathbf{E}$ into the relatively compact subsets of $\mathbf{E}$.

3.2. **Theorem.** *Let* $\mathbf{E}$ *be an AL-space or a Dedekind complete AM-space with unit. Then the following are equivalent*:

(a) $\mathbf{E}$ *contains no discrete elements.*
(b) $I \wedge |T| = 0$ *holds for every Dunford-Pettis operator* $T\colon \mathbf{E} \to \mathbf{E}$.
(c) *Daugavet's equation holds for every Dunford-Pettis operator* $\mathbf{E} \to \mathbf{E}$.
(d) $I \wedge |T| = 0$ *holds for every linear operator* $T\colon \mathbf{E} \to \mathbf{E}$ *of rank one.*
(e) *Daugavet's equation holds for every linear operator* $\mathbf{E} \to \mathbf{E}$ *of rank one.*

Proof. Assume first that $\mathbf{E}$ contains no discrete elements. Consider a Dunford-Pettis operator $T\colon \mathbf{E} \to \mathbf{E}$ and define $S := I \wedge |T|$. Then S is an orthomorphism [3, Theorem 8.11].

If $\mathbf{E}$ is an AL-space, then the Dunford-Pettis operators and the AM-compact operators $\mathbf{E} \to \mathbf{E}$ are the same and form a band of $\mathscr{L}(\mathbf{E})$ [3, Theorem 19.18]; thus, S is a positive AM-compact orthomorphism, and Lemma 3.1 yields $S = 0$.

If $\mathbf{E}$ is a Dedekind complete AM-space with unit, then the Dunford-Pettis operators and the weakly compact operators $\mathbf{E} \to \mathbf{E}$ are the same [3, Theorems 19.6, 19.4, and 19.23] and form an ideal of $\mathscr{L}(\mathbf{E})$, by [15] and [3, Theorem 17.10]; thus, S is a positive weakly compact orthomorphism, hence S^2 is a positive compact orthomorphism [3, Corollary 19.9], Lemma 3.1 yields $S^2 = 0$, and this implies $S = 0$, by [3, Theorem 8.18].

Therefore, (a) implies (b).

Assume now that $\mathbf{E}$ contains a discrete element $u \in \mathbf{E}_+$. Then the ideal $I(\{u\})$ generated by $\{u\}$ in $\mathbf{E}$ is a projection band [2, Theorem 2.16], the band projection $P: \mathbf{E} \to I(\{u\})$ has rank one, and Daugavet's equation fails for $T := -P$.

Therefore, (e) implies (a).

The remaining implications are obvious from Lemma 2.3. $\square$

In the case where $\mathbf{E}$ is an AL-space, the assertion of Theorem 3.2 is a proper extension of a result of Holub [10]; in the case where $\mathbf{E}$ is a Dedekind complete AM-space with unit, it is essentially contained in the results of Foias and Singer [8] and Chauveheid [5].

If $\mathbf{E}$ is either an AL-space or a Dedekind complete AM-space with unit, then a linear operator $\mathbf{E} \to \mathbf{E}$ is an *almost integral* operator if it is contained in the band generated by the linear operators of rank one in $\mathscr{L}(\mathbf{E})$.

3.3. **Corollary.** *Let $\mathbf{E}$ be an AL-space or a Dedekind complete AM-space with unit. Then the following are equivalent:*

(a) *$\mathbf{E}$ contains no discrete elements.*
(b) *$I \wedge |T| = 0$ holds for every almost integral operator $T: \mathbf{E} \to \mathbf{E}$.*
(c) *Daugavet's equation holds for every almost integral operator $\mathbf{E} \to \mathbf{E}$.*

Corollary 3.3 is essentially due to Lozanovskii [13] and Synnatzschke [16, 17]; see also Synnatzschke [18] for an application to an AL-space of linear operators.

4. Remarks

The fact that Daugavet's equation fails for $T := -I$ can be generalized as follows: If $T: \mathbf{E} \to \mathbf{E}$ is a linear operator satisfying $0 < T \leq I$, then $\|I - T\|$ is strictly smaller than $1 + \|T\|$.

The following result can be proven in the same way as Lemmas 2.1 and 2.3:

4.1. **Lemma.** *Let $\mathbf{E}$ be an AL-space. If $J: \mathbf{E} \to \mathbf{E}$ is a positive isometry, then*

$$\|J + T\| = 1 + \|T\|$$

holds for every positive operator $T: \mathbf{E} \to \mathbf{E}$ and for each $T \in \mathscr{L}(\mathbf{E})$ satisfying $J \wedge |T| = 0$.

A corresponding result holds in the case where $\mathbf{E}$ is an AM-space or a Dedekind complete AM-space with unit, respectively, if in Lemmas 2.1 and 2.3 the identity operator is replaced by a positive operator which is surjective.

Let us finally remark that all results of this paper apply to the case $\mathbf{E} := L^p(\mu)$ with $p \in \{1, \infty\}$, and that $L^p(\mu)$ contains no discrete elements if and only if μ is nonatomic. In particular, Theorem 3.2 corrects a statement of Chauveheid [5] and Holub [9] concerning $L^\infty(\mu)$, and Corollary 3.3 applies to absolute kernel operators on $L^1(\mu)$ or $L^\infty(\mu)$; see [19, Chapter 13].

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References

1. Y. A. Abramovich, *A new proof of a theorem of J. Holub*, preprint (1988).
2. C. D. Aliprantis and O. Burkinshaw, *Locally solid Riesz spaces*, Academic Press, New York and London 1978.
3. ——, *Positive operators*, Academic Press, New York and London 1985.
4. V. F. Babenko and S. A. Pichugov, *A property of compact operators in the space of integrable functions*, Ukrainian Math. J. **33** (1981), 374–376.
5. P. Chauveheid, *On a property of compact operators in Banach spaces*, Bull. Soc. Roy. Sci. Liège **51** (1982), 371–378.
6. I. K. Daugavet, *On a property of compact operators in the space C* (Russian), Uspekhi Mat. Nauk **18** (1963), 157–158.
7. U. U. Diallo and P. P. Zabreiko, *On the Daugavet-Krasnoselskii theorem* (Russian), Vestsi Akad. Navuk BSSR Ser. Fiz.-Mat. Navuk (1987), 26–31.
8. C. Foias and I. Singer, *Points of diffusion of linear operators and almost diffuse operators in spaces of continuous functions*, Math. Z. **87** (1965), 434–450.
9. J. R. Holub, *A property of weakly compact operators on $C[0,1]$*, Proc. Amer. Math. Soc. **97** (1986), 396–398.
10. ——, *Daugavet's equation and operators on $L^1(\mu)$*, Proc. Amer. Math. Soc. **100** (1987), 295–300.
11. H. Kamowitz, *A property of compact operators*, Proc. Amer. Math. Soc. **91** (1984), 231–236.
12. M. A. Krasnoselskii, *A class of linear operators in the space of abstract continuous functions*, Math. Notes **2** (1967), 856–858.
13. G. Y. Lozanovskii, *On almost integral operators in KB-spaces* (Russian), Vestnik Leningrad Univ. (1966), no. 7, 35–44.
14. H. H. Schaefer, *Banach lattices and positive operators*, Springer, Berlin and New York 1974.
15. K. D. Schmidt, *On the modulus of weakly compact operators and strongly additive vector measures*, Proc. Amer. Math. Soc. **102** (1988), 862–866.
16. J. Synnatzschke, *On almost integral operators in K-spaces*, Vestnik Leningrad Univ. Math. **4** (1977), 243–252.

17. ——, *On the adjoint of a regular operator and some of its applications to the question of complete continuity and weak complete continuity of regular operators*, Vestnik Leningrad Univ. Math. **5** (1978), 71–81.

18. ——, *Über eine additive Normgleichheit für Operatoren in Banachverbänden*, Math. Nachr. **117** (1984), 175–180.

19. A. C. Zaanen, *Riesz spaces* II, North-Holland, Amsterdam and New York 1983.

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