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THE HEREDITARY DUNFORD–PETTIS PROPERTY FOR $l_1(E)$

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ABSTRACT. A Banach space E is said to be hereditarily Dunford–Pettis if all of its closed subspaces have the Dunford–Pettis property. In this note we prove that the Banach space $l_1(E)$, of all absolutely summing sequences in E with the usual norm, is hereditarily Dunford–Pettis if and only if E is also.

Given a Banach space E we denote by $l_1(E)$ the Banach space of all absolutely summing sequences $x = (x_n)$ in E endowed with the norm

$$\|x\| = \sum_{n=1}^{\infty} \|x_n\| .$$

A Banach space E is said to have the Dunford–Pettis property if for every pair of weakly null sequences $(x_n) \subset E$ and $(x'_n) \subset E'$ one has $\lim\langle x_n, x'_n\rangle = 0$. Following Diestel [2] we say that E is hereditarily Dunford–Pettis (or also that E has the hereditary Dunford–Pettis property) if all its closed subspaces have the Dunford–Pettis property. Examples of spaces enjoying this property are the classical Banach spaces c_0, l_1, and a certain class of $C(K)$-spaces (see [2]). It has been studied recently (see [1] and [4]) in which conditions the spaces $c_0(E)$ and $C(K, E)$ (the vectorial version of c_0 and $C(K)$) are hereditarily Dunford–Pettis. The aim of this note is to characterize when $l_1(E)$ has the cited property. We shall prove that $l_1(E)$ is hereditarily Dunford–Pettis if and only if E is also.

The notations and terminology used here can be found in [2, 3, 5].

It is clear (see [3]) that a normalized basic sequence (x_n) in a Banach space is equivalent to the unit vector basis of c_0 if and only if there is a constant $C > 0$ such that

$$\left\|\sum_{i=1}^{n} a_i x_i\right\| \le C \sup_{1\le i\le n} |a_i|$$

holds for any $n \in \mathbf{N}$ and any scalars $a_1, a_2, \dots, a_n$. In this case we say that (x_n) is C-equivalent to the unit vector basis of c_0.

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To prove our theorem we need some known results. The proof of the first one is analogous to the scalar case. The two following ones give two different characterizations of the hereditary Dunford–Pettis property.

Lemma 1. *A bounded sequence* $(x^n) = ((x_i^n)_i) \subset l_1(E)$ *is weakly convergent to zero if and only if the following two conditions are satisfied*:

(1) $(x_i^n)_n$ *is weakly convergent to zero in* E *for all* $i \in \mathbf{N}$.

(2) *For each* $\varepsilon > 0$ *there is* $i_\varepsilon \in \mathbf{N}$ *such that* $\sum_{i=i_\varepsilon}^{\infty} \|x_i^n\| < \varepsilon$ *for all* $n \in \mathbf{N}$.

Proposition 2 [1]. *A Banach space* E *is hereditarily Dunford–Pettis if and only if every normalized weakly null sequence in* E *has a subsequence that is equivalent to the unit vector basis of* c_0.

Theorem 3 [4]. *A Banach space* E *is hereditarily Dunford–Pettis if and only if there is a constant* $C > 0$ *such that every normalized weakly null sequence in* E *has a subsequence that is* C*-equivalent to the unit vector basis of* c_0.

Now we can prove the announced result:

Theorem. $l_1(E)$ *is hereditarily Dunford–Pettis if and only if* E *is also.*

Proof. The necessity is clear because E is isomorphic to a subspace of $l_1(E)$. For the converse suppose that E is hereditarily Dunford–Pettis. Let $C > 0$ be the constant that appears in Theorem 3. Let $(x^n) = ((x_i^n)_i) \subset l_1(E)$ be a normalized weakly null sequence. According to Proposition 2 we must prove that (x^n) has a subsequence that is equivalent to the unit vector basis of c_0. By Lemma 1 there exists at least one positive integer i such that $\|x_i^n\| \underset{n}{\not\to} 0$ (otherwise $\|x^n\| \to 0$ in $l_1(E)$). Let i_1 be the first positive integer such that $\|x_{i_1}^n\| \underset{n}{\not\to} 0$. Then by Theorem 3 there is a subsequence $(x_{i_1}^{\sigma_1(n)})_n$ of $(x_{i_1}^n)_n$ such that $\inf_n \|x_{i_1}^{\sigma_1(n)}\| > 0$ and

$$\left\| \sum_n a_n x_{i_1}^{\sigma_1(n)} \right\| \le C \sup_n |a_n| \sup_n \left\| x_{i_1}^{\sigma_1(n)} \right\|$$

for all finite sequence (a_n) of scalars.

By passing to a subsequence if necessary, we can also assume that $\lim_n \|x_{i_1}^{\sigma_1(n)}\| > 0$ exists. In case that $(x_j^{\sigma_1(n)})_n$ does not converge to zero for all $j > i_1$, let i_2 be the first positive integer, $i_2 > i_1$, such that $\|x_{i_2}^{\sigma_1(n)}\| \underset{n}{\not\to} 0$. Now, by the preceding argument, it follows that there is a subsequence $(x_{i_2}^{\sigma_2(n)})_n$ of $(x_{i_2}^{\sigma_1(n)})_n$ such that

$$\left\| \sum_n a_n x_{i_2}^{\sigma_2(n)} \right\| \le C \sup_n |a_n| \sup_n \left\| x_{i_2}^{\sigma_2(n)} \right\|$$

for all finite sequence (a_n) of scalars, and for which $\lim_n \|x_{i_2}^{\sigma_2(n)}\| > 0$ exists. Then, by induction, we obtain an increasing sequence $(i_k)_{k \in J}$ in $\mathbf{N}$ (finite or infinite) and a family of sequences in E, $\{(x_{i_k}^{\sigma_k(n)})_n : k \in J\}$, verifying the following conditions:

(1) $(x_{i_k}^{\sigma_k(n)})_n$ is a subsequence of $(x_{i_k}^{\sigma_{k-1}(n)})_n$ for all $k \in J$ (where $\sigma_0 : \mathbf{N} \to \mathbf{N}$ is the identity map),

(2) $\|\sum_n a_n x_{i_k}^{\sigma_k(n)}\| \le C \sup_n |a_n| \sup_n \|x_{i_k}^{\sigma_k(n)}\|$ for all finite sequences (a_n) of scalars, and all $k \in J$,

(3) $\lim_n \|x_{i_k}^{\sigma_k(n)}\| > 0$ exists for all $k \in J$, and

(4) $\lim_n \|x_j^{\sigma_k(n)}\| = 0$ for $j \notin \{i_k : k \in J\}$.

Let us consider the following sequence (y^n): if J is finite and $l = \max J$ we put $(y^n) = (x^{\sigma_l(n)})_n$; if J is infinite we put $(y^n) = (x^{\sigma_n(n)})_n$. In any case (y^n) is a subsequence of (x^n) verifying:

(a) $\|\sum_n a_n y_j^n\| \le C \sup_n |a_n| \sup_n \|y_j^n\|$ for all finite sequence (a_n) of scalars and all $j \in I = \{i_k : k \in J\}$.

(b) $\lim_n \|y_j^n\| = \delta_j > 0$ exists for all $j \in I$.

(c) $\lim_n \|y_j^n\| = 0$ for all $j \notin I$.

By a standard diagonalization process in case that $\mathbf{N}\backslash I$ is infinite, or by passing simply to a subsequence if $\mathbf{N}\backslash I$ is finite, we can assume that the sequence (y^n) also verifies

(c′) $\sum_{n=1}^{\infty} \|y_j^n\| < \frac{1}{2^j}$ for all $j \notin I$.

Now, since $\lim_n \|y_j^n\| = \delta_j > 0$ for all $j \in I$, using again the preceding argument, we get a subsequence (z^n) of (y^n) such that

(d) $\delta_j - \frac{1}{2^j} < \|z_j^n\| < \delta_j + \frac{1}{2^j}$ for all $n \in \mathbf{N}$ and all $j \in I$.

Since $z^n \in l_1(E)$, then $\sum_{j \in I} \|z_j^n\| \le \sum_{j \in \mathbf{N}} \|z_j^n\| < +\infty$. And therefore, according to (d), we have that $\sum_{j \in I} \delta_j < +\infty$. Thus

(e) $\sum_{j \in I} \sup_n \|z_j^n\| \le \sum_{j \in I} (\delta_j + \frac{1}{2^j}) = M < +\infty$.

Finally, by the Bessaga–Pelczynski selection principle (see [3]), we may assume that (z^n) is a basic sequence. Now, we claim that (z^n) is equivalent to the unit vector basis of c_0. To prove this choose $r \in \mathbf{N}$ and a finite sequence $(a_n)_{n=1}^r$ of scalars, then by (a), (c′), and (e) we have

$$\begin{aligned}
\left\|\sum_{n=1}^{r} a_n z^n\right\| \le \sum_{j=1}^{\infty} \left\|\sum_{n=1}^{r} a_n z_j^n\right\| &= \sum_{j \in I} \left\|\sum_{n=1}^{r} a_n z_j^n\right\| + \sum_{j \notin I} \left\|\sum_{n=1}^{r} a_n z_j^n\right\| \\
&\le C \sup_{1 \le n \le r} |a_n| \sum_{j \in I} \sup_n \|z_j^n\| + \sup_{1 \le n \le r} |a_n| \sum_{j \notin I} \sum_{n=1}^{r} \|z_j^n\| \\
&\le CM \sup_{1 \le n \le r} |a_n| + \sup_{1 \le n \le r} |a_n| \sum_{j \notin I} \frac{1}{2^j} \\
&\le (CM + 1) \sup_{1 \le n \le r} |a_n| .
\end{aligned}$$

Hence (z^n) is $(CM+1)$-equivalent to the unit vector basis of c_0. This finishes the proof.

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