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TWO GENERALIZATIONS OF TITCHMARSH'S CONVOLUTION THEOREM

RAOUF DOSS

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ABSTRACT. Titchmarsh's convolution theorem states that if the functions f, g vanish on $(-\infty, 0)$ and if the convolution $f * g(t) = 0$ on an interval $(0, T)$, then there are two numbers α, $\beta \geq 0$ such that $\alpha + \beta = T$, $f = 0$ a.e. on $(0, \alpha)$, and $g = 0$ a.e. on $(0, \beta)$. T may be infinite. For the case $T = \infty$ we prove that if $f * g = 0$ on R and one of the two functions f, g is 0 on $(-\infty, 0)$, then either f or g is 0 a.e. on R. Next we consider the integro-differential-difference equation $f * g(t) + \sum \lambda_{p\sigma} f^{(p)}(t - a_{p\sigma}) = 0$ for t in $(0, T)$, where $a_{p\sigma} \geq 0$, $\lambda_{p\sigma}$ are constants. Conclusions similar to Titchmarsh's hold with the additional information that $\alpha \geq T - a_{p\sigma}$ whenever $\lambda_{p\sigma} \neq 0$.

For f, $g \in L^1(R)$ the convolution $f * g$ is defined as

$$f * g(t) = \int_R f(t - x) g(x)\, dx.$$

Titchmarsh's theorem states that if f, $g = 0$ on the interval $(-\infty, 0)$ and if

$$f * g(t) = 0 \qquad \text{for } t \in (0, T),$$

then there are numbers α, $\beta \geq 0$ with $\alpha + \beta = T$ for which $f(x) = 0$ for almost all x in $(0, \alpha)$ and $g(x) = 0$ for almost all x in $(0, \beta)$. T may be infinite.

There are many different proofs of this famous theorem; most of them, like Titchmarsh's [11], Crum [2], Dufresnoy [4], Boas [1], Koosis [7], and Lax [8], are based on the theory of analytic or harmonic functions; others, like Mikusinski [10], use real variable methods; still others, like Helson [5], Doss [3], rely on harmonic analysis. For an extension to functions of several variables see Lions [9] and to functions taking values in certain Banach algebras see Mikusinski [10].

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Our first generalization of Titchmarsh's theorem deals with the case $T = \infty$. We shall prove

Theorem I. *Suppose that f, g are integrable on R, that either f or g vanishes on $(-\infty, 0)$ and that $f * g = 0$ on R. Then either $f = 0$ or $g = 0$* a.e. *on R.*

Observe that, for $T = \infty$, Titchmarsh's hypotheses also include $f * g = 0$ on R.

It is well known that some restriction on the behavior of f or g is needed if we want the relation $f * g = 0$ to imply that either f or g is 0 a.e. on R. For example, take an $f \in L^1(R)$, whose Fourier transform $\hat{f}$ has compact support K. The function $g(x) = e^{i\lambda x} f(x)$ will have a transform $\hat{g}$ whose support is $K + \lambda$; if λ is large enough $\hat{f}\hat{g}$ will be zero everywhere so that $f * g = 0$ a.e. on R. Nevertheless, none of the functions f or g is 0 a.e. on R.

Our second generalization deals with an integro-differential-difference equation on a finite interval.

Theorem II. *Let f, $g \in L^1(-\infty, \infty)$, f, $g = 0$ on the interval $(-\infty, 0)$ and consider the integro-differential-difference equation*

$$\int_0^y g(y-x) f(x)\, dx + \sum_{\rho=0}^{r} \sum_{\sigma=0}^{s} \lambda_{\rho\sigma} f^{(\rho)}(y - a_{\rho\sigma}) = 0 \tag{1}$$

for $y \in (0, T)$, where $\lambda_{\rho\sigma}$ are constants and for each ρ, $a_{\rho\sigma}$ are distinct non-negative constants. Then there are two numbers α, $\beta \geq 0$ such that

$$\alpha + \beta = T,$$

$$\begin{aligned} f(x) &= 0 \quad \text{a.e. } on\ (0, \alpha), \\ g(x) &= 0 \quad \text{a.e. } on\ (0, \beta), \end{aligned}$$

and $\alpha \geq T - a_{\rho\sigma}$ whenever $\lambda_{\rho\sigma} \neq 0$.

Remarks. It is implicitly assumed that the derivatives $f^{(\rho)}$ appearing in (1) exist *everywhere* on $(0, T)$.

Proof of Theorem I. Let f, g be as in the hypotheses of Theorem I. We may suppose that f vanishes on $(-\infty, 0)$. Then the Fourier transform

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-isx} f(x)\, dx$$

extends to a continuous function on the closed half-plane $\{\operatorname{Im} s \leq 0\}$ which is analytic in $\{\operatorname{Im} s < 0\}$. Therefore, if $\hat{f} = 0$ on some nonempty open interval, then $\hat{f} = 0$ identically, so $f = 0$ a.e. [6].

Now suppose g is a nonzero function. Then $\hat{g}$ is nonvanishing on some nonempty open interval I. Since $\hat{f}\hat{g} = 0$ identically by the hypotheses, it follows that $\hat{f} = 0$ on I. Hence $f = 0$ a.e. as desired.

Remark. In Theorem I the vanishing interval $(-\infty, 0)$ for f or g can be replaced by an interval of the form $(-\infty, a)$, or (a, ∞), where a is any real number.

Proof of Theorem II. We may assume $a_{\rho\sigma} \leq T$, for if $T < a_{\rho\sigma}$ then the corresponding term $\lambda_{\rho\sigma} f^{(\rho)}(y - a_{\rho\sigma})$ is 0 on the considered interval $(0, T)$ while the relation $\alpha \geq T - a_{\rho\sigma}$ is redundant since we already have $\alpha \geq 0$.

Now let $(-\infty, \alpha)$ be the largest interval on which f vanishes almost everywhere, so that $\alpha \geq 0$. In case $\alpha \geq T$, the desired conclusions hold with α replaced by T and $\beta = 0$. Therefore we may suppose $\alpha < T$.

Particular case: $r = 0$. The equation is

$$\int_0^y g(y-x) f(x)\, dx + \sum_{\sigma=0}^{s} \lambda_\sigma f(y - a_\sigma) = 0 \qquad \text{for } y \in (0, T). \tag{2}$$

Let χ^δ be the characteristic function of the interval $(0, \delta)$, where $\delta > 0$ is so small that the intervals $(a_\sigma, a_\sigma + \delta)$ do not overlap. For $a \geq 0$ and real numbers t, y, we have

$$\chi^\delta(t - a - y) = 1 \qquad \text{iff } y \in (t - a - \delta, t - a).$$

Hence

$$\begin{aligned} \int_{t-\delta}^{t} \lambda f(y-a)\, dy &= \int_{t-a-\delta}^{t-a} \lambda f(y)\, dy \\ &= \int_{-\infty}^{t} \lambda \chi^\delta(t-a-y) f(y)\, dy = \int_0^t \lambda \chi_a^\delta(t-y) f(y)\, dy, \end{aligned} \tag{3}$$

where χ_a^δ is the a-translate of χ^δ: $\chi_a^\delta(u) = \chi^\delta(u - a)$.

Next, put

$$G(t) = \int_0^t g(y)\, dy \qquad \text{and} \qquad G^\delta(t) = G(t) - G(t - \delta).$$

We have, by Fubini's theorem

$$\begin{aligned} \int_0^t \int_0^y g(y-x) f(x)\, dx\, dy &= \int_0^t \left(\int_x^t g(y-x)\, dy \right) f(x)\, dx \\ &= \int_0^t G(t-x) f(x)\, dx. \end{aligned}$$

Similarly

$$\int_0^{t-\delta} \int_0^y g(y-x) f(x)\, dx\, dy = \int_0^t G(t - \delta - x) f(x)\, dx.$$

Hence

$$\int_{t-\delta}^{t} \int_0^y g(y-x) f(x)\, dx\, dy = \int_0^t G^\delta(t-x) f(x)\, dx. \tag{4}$$

Observe that (2) is valid even for $y < 0$. Integrating (2) with respect to y on $(t-\delta, t)$, we get, using (4) and relations similar to (3)

$$\int_0^t \left[G^\delta(t-x) + \sum_{\sigma=0}^{s} \lambda_\sigma \chi_{a_\sigma}^\delta(t-x)\right] f(x)\,dx = 0 \qquad \text{for } t \in (0, T). \tag{5}$$

Since $G^\delta(x) = 0$ and $\chi_{a_\sigma}^\delta(x) = 0$ for $x < 0$, we deduce from (5) and Titchmarsh's convolution theorem that there are two numbers α', $\beta' \geq 0$ such that

$$\alpha' + \beta' = T,$$
$$f(x) = 0 \qquad \text{a.e. on } (0, \alpha'), \text{ and}$$
$$G^\delta(x) + \sum_{\sigma=0}^{s} \lambda_\sigma \chi_{a_\sigma}^\delta(x) = 0 \qquad \text{a.e. on } (0, \beta').$$

Then $\alpha' \leq \alpha$, $\beta' = T - \alpha' \geq T - \alpha = \beta$ (say). Thus we have

$$\begin{gathered} \alpha + \beta = T, \\ f(x) = 0 \qquad \text{a.e. on } (0, \alpha), \\ G^\delta(x) + \sum_{\sigma=0}^{s} \lambda_\sigma \chi_{a_\sigma}^\delta(x) = 0 \qquad \text{a.e. on } (0, \beta), \end{gathered} \tag{6}$$

and (6) is valid for any $\delta > 0$. Multiplying (6) by δ^{-1} and making δ tend to 0 we get

$$g(x) = 0 \qquad \text{a.e. on } (0, \beta).$$

Now assume $\beta > a_\sigma$, $\lambda_\sigma \neq 0$ for some particular $\sigma = 0, 1, \ldots, s$. Then, by (6) we have

$$G^\delta(x) + \lambda_\sigma \chi_{a_\sigma}^\delta(x) = 0$$

in some neighborhood of a_σ. Hence

$$\lim_{x \to a_\sigma^-} G^\delta(x) = 0 \qquad \text{and} \qquad \lim_{x \to a_\sigma^+} G^\delta(x) = -\lambda_\sigma.$$

This is impossible since G^δ is continuous. Thus $\lambda_\sigma \neq 0$ implies $\beta \leq a_\sigma$, that is $T - \alpha \leq \alpha_\sigma$ and finally $\alpha \geq T - a_\sigma$.

The case $r = 0$ is now proved.

General case: $r > 0$. The proof is by induction on r. So suppose that Theorem II is true for $r - 1$.

Integrating (1) with respect to y on $(t - \delta, t)$ we get

$$\int_0^t \left[G^\delta(t-x) + \sum_{\sigma=0}^{s} \lambda_{0\sigma} \chi_{a_{0\sigma}}^\delta(t-x)\right] f(x)\,dx$$
$$+ \sum_{\rho=1}^{r} \sum_{\sigma=0}^{s} \lambda_{\rho\sigma}[f^{(\rho-1)}(t - a_{\rho\sigma}) - f^{(\rho-1)}(t - \delta - a_{\rho\sigma})] = 0$$

for $t \in (0, T)$. By the induction hypothesis and the proof for the case $r = 0$, we have

$$\begin{aligned} f &= 0 \qquad \text{a.e. on } (0, \alpha), \\ g &= 0 \qquad \text{a.e. on } (0, T - \alpha), \text{ and} \end{aligned} \tag{7}$$

$$\alpha \geq T - a_{\rho\sigma} \text{ whenever } \lambda_{\rho\sigma} \neq 0, \qquad \rho = 0, \ldots, r, \quad \sigma = 0, \ldots, s.$$

This completes the induction process.

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c/o Louis Pigno, Department of Mathematics, Kansas State University, Manhattan, Kansas 66506