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GLOBAL C^r STRUCTURAL STABILITY OF VECTOR FIELDS ON OPEN SURFACES WITH FINITE GENUS

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ABSTRACT. A vector field X on the open manifold M is globally C^r structurally stable if X has a neighborhood U in the Whitney C^r topology such that the trajectories of every vector field $Y \in \mathsf{U}$ can be mapped onto trajectories of X by a homeomorphism $h: M \to M$ which is in a preassigned compact-open neighborhood of the identity. In [2] it was proved the theorem formulating the sufficient conditions for global C^r $(r \geq 1)$ structural stability of vector fields on open surfaces $(\dim M = 2)$. These conditions are also necessary for global C^r structural stability on the plane if $r \geq 1$ (see [2]) and for $r = 1$ on any open surface of finite genus [1]. Here we will generalize it for C^r $(r \geq 1)$ vector fields defined on open orientable surface with finite genus and countable space of ends E.

DEFINITIONS AND STATEMENT OF THE RESULT

Let M be an open orientable surface with finite genus and countable space of ends E (also called the ideal boundary of M). A boundary component of surface M is a nested sequence $P_1 \supset P_2 \supset \cdots$ of connected unbounded regions in M such that:

(a) the boundary of P_n in M is compact for all n,

(b) for any boundary subset K of M, $P_n \cap K = \varnothing$ for n sufficiently large.

We say that two boundary components $P_1 \supset P_2 \supset \cdots$ and $P_1' \supset P_2' \supset \cdots$ are equivalent if for any n there is a corresponding integer m such that $P_m \subset P_n'$ and vice versa. Let P^* denote the equivalence class of boundary components containing $P_1 \supset P_2 \supset \cdots$. The ideal boundary E of a surface M is the topological space having the equivalence classes of boundary components of M as elements and endowed with such a topology that E with it is homeomorphic to a subset of a Cantor set.

By $H^r(M)$ we denote the space of complete C^r vector fields on M with the C^r Whitney topology $(r \geq 1)$. X, Y denote elements of $H^r(M)$, ϕ_X denotes the flow induced by X. For $x \in M$, $O_X(x)$ $(O_X^+(x), O_X^-(x))$ is the trajectory of x (the positive semitrajectory, the negative semitrajectory) under ϕ_X. By $O_X[x, y]$ we denote the closed X-trajectory segment from x to y.

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We distinguish three kinds of asymptotic behavior for each semitrajectory $O_X^{\pm}(x)$:

(a) $O_X^{\pm}(x)$ is bounded if it is contained in some compact set $K \subset M$,

(b) $O_X^{\pm}(x)$ escapes to infinity if for each compact set $K \subset M$ there is a point $y \in O_X^{\pm}(x)$ for which $O_X^{\pm}(y) \cap K = \varnothing$,

(c) $O_X^{\pm}(x)$ oscillates if it is neither bounded nor escapes to infinity.

These kinds of behavior for $O_X^{+}(x)$ (resp. $O_X^{-}(x)$) also can be described in terms of the ω-limit (resp. α-limit) set of $x \in M$ under ϕ_X:

$$\omega(O_X^+(x)) = \{y \in M : \exists t_n \to +\infty \quad \text{such that } \phi_X(x, t_n) \to y\},$$
$$\alpha(O_X^-(x)) = \{y \in M : \exists t_n \to -\infty \quad \text{such that } \phi_X(x, t_n) \to y\}.$$

It is easy to see that $O_X^+(x)$

(a) is bounded iff $\omega(O_X^+(x))$ is compact (and nonempty),

(b) escapes to infinity iff $\omega(O_X^+(x)) = \varnothing$,

(c) oscillates iff $\omega(O_X^+(x))$ is a noncompact subset of M.

We extend the definition of ω-limit (resp. α-limit) set of $x \in M$ to ω^*-limit (resp. α^*-limit) set:

$$\omega^*(O_X^+(x)) = \{y \in M \cup E : \exists t_n \to +\infty \quad \text{such that } \phi_X(x, t_n) \to y\},$$
$$\alpha^*(O_X^-(x)) = \{y \in M \cup E : \exists t_n \to -\infty \quad \text{such that } \phi_X(x, t_n) \to y\}.$$

Thus

(a) $O_X^+(x)$ escapes to infinity iff there is $P^* \in E$ such that $\omega^*(O_X^+(x)) = P^*$,

(b) $O_X^+(x)$ oscillates iff $\omega(O_X^+(x)) \neq \varnothing$ and there is $P^* \in E$ such that $P^* \in \omega^*(O_X^+(x))$.

Let $\mathrm{Per}(X)$, $\Omega(X)$ denote, respectively, the set of periodic points, the set of nonwandering points of X, i.e.:

$$\mathrm{Per}(X) = \{x \in M : \phi_X(x, t) = x \quad \text{for some } t > 0\},$$

$$\Omega(X) = \{x \in M : \exists x_n \to x, t_n \to +\infty \quad \text{such that } \phi_X(x_n, t_n) \to x\}.$$

We define the first positive (resp. negative) prolongation limit set $x \in M$ by:

$$J_X^{\pm}(x) = \{y \in M : \exists x_n \to x, t_n \to \pm\infty \quad \text{such that } \phi_X(x_n, t_n) \to y\}.$$

In general $\mathrm{Per}(X) \subset \Omega(X)$, $\omega(O_X^+(x)) \subset J_X^+(x)$, $\alpha(O_X^-(x)) \subset J_X^-(x)$ and $\Omega(X) = \{x \in M : x \in J_X^+(x)\}$.

Modifying the definition of Nemytskii–Stepanov [4], we say two unbounded semitrajectories $O_X^+(x)$ and $O_X^-(x)$ form a saddle at infinity if $y \in J_X^+(x)$ and $O_X^+(x)$, $O_X^-(y)$ escape to infinity (i.e. there are points P^*, $Q^* \in E$ such that $\omega^*(O_X^+(x)) = P^*$, $\alpha^*(O_X^-(y)) = Q^*$). In this case, we call $O_X^+(x)$ (resp. $O_X^-(y)$) the stable (resp. unstable) separatrix of the saddle at infinity.

By W_X^+ (resp. W_X^-) we denote the union of all stable (resp. unstable) separatrices of fixed saddles and saddles at infinity. Each set is ϕ_X invariant; it may consist of finitely or infinitely many distinct trajectories. In either case, it is not generally closed, since a fixed saddle belongs to the closure of its separatrices.

By a transverse section S to a vector field $X \in H^r(M)$ we mean an embedded interval. A flowbox for $X \in H^r(M)$ is a closed quadrilateral $F \subset M$ containing no critical points of X, with two opposite edges $S^{\pm}$ transverse to X and the other two edges X-trajectory segments, each joining an endpoint of S^+ to an endpoint of S^-. We call S^+ the entrance set and S^- the exit set of F.

Let $H_0^r(M)$ be the set of vector fields $X \in H^r(M)$ with hyperbolic critical points, $H_{K-S}^r(M)$ be the set of Kupka-Smale vector fields, i.e. $X \in H_{K-S}^r(M)$ if $X \in H^r(M)$ and satisfies:

(a) for each $x \in \mathrm{Per}(X)$ the trajectory $O_X(x)$ is hyperbolic,

(b) there are no saddle connections between fixed saddles.

In [5] it was proved that $H_0^r(M)$ is open and dense in $H^r(M)$ while $H_{K-S}^r(M)$ is residual in $H^r(M)$.

Recall that a minimal set is a nonempty compact invariant set with no proper compact invariant subsets. Trivial minimal sets are trajectories in $\mathrm{Per}(X)$.

With these definitions, we can formulate sufficient conditions for global structural stability proved in [2].

Theorem A. *Let M be an open surface. If $X \in H^r(M)$ $(r \geq 1)$ is a vector field satisfying:*

(a) *every trajectory in* $\mathrm{Per}(X)$ *is hyperbolic,*

(b) *X has no nontrivial minimal sets and no oscillating trajectories,*

(c) $\mathrm{cl}\, W_X^+ \cap \mathrm{cl}\, W_X^- \subset \mathrm{Per}(X)$

then $\Omega(X) = \mathrm{Per}(X)$ *and X is globally C^r structurally stable.*

Theorem B. *Let $X \in H^r(M)$ be globally C^r structurally stable vector field. Then conditions of Theorem* A (a)–(c) *hold if*:

(a) *$M = R^2$ and $r \geq 1$,*

(b) *M is any open surface of finite genus and $r = 1$.*

Part (a) *is proved in* [2], *part* (b) *in* [1].

In the next section, we will prove the following:

Theorem. *Let M be an open orientable surface with finite genus and countable space of ends. Then conditions of Theorem* A (a)–(c) *are necessary for global C^r $(r \geq 1)$ structural stability of vector fields on M.*

Proof of the theorem. We start with the theorem proved in [2].

Proposition 1. *Let $X \in H^r(M)$ be globally C^r structurally stable vector field, $r \geq 1$. Then every trajectory in* $\mathrm{Per}(X)$ *is hyperbolic.*

Corollary 1. *If $X \in H^r(M)$ is globally C^r $(r \geq 1)$ structurally stable vector field then $X \in H_{K-S}^r(M)$.*

Proof. X is topologically conjugated to some vector field $Y \in H_{K-S}^r(M)$ since $H_{K-S}^r(M)$ is a residual subset of $H^r(M)$. By Proposition 1 every trajectory in

$\operatorname{Per}(X)$ is hyperbolic. As Y has no connection between fixed saddles and this property is preserved by conjugacy, it implies that $X \in H^r_{K-S}(M)$.

Proposition 2. *A globally C^r $(r \geq 1)$ structurally stable vector field has no nontrivial minimal sets.*

Proof. Suppose that K is a nontrivial minimal set of X. Let U be a neighborhood of X in $H^r(M)$ such that for each $Y \in \mathsf{U}$ there is a homeomorphism h_Y of M conjugating X with Y. Let $Z \in \mathsf{U}$ be C^∞ vector field. Thus $h_Z(K)$ is a nontrivial minimal set of Z. We choose a neighborhood U of K with compact closure. Richards [6] proved that for any open surface M with finite genus g and space of ends E there is a C^∞ diffeomorphism $f\colon M \to N - E'$ where N is a compact surface of genus g and E' is a closed, totally disconnected subset of N. Thus $Z' = Df(Z)$ is a vector field of class C^∞ defined on $f(M)$ and $K' = f(h_Z(K))$ is contained in $U' = f(U) \subsetneq N$. Applying a smooth partition of unity we may extend $Z'_{|K'}$ to C^∞ vector field Z_1 defined on N such that $Z_1(y) = Z'(y)$ for $y \in K'$, $Z_1(y) = 0$ for $y \notin U'$. It implies that K' is a nontrivial minimal set of Z_1. By Schwartz's theorem [7] any nontrivial minimal set of C^2 vector field on N is the whole surface and N is a two-dimensional torus which contradicts the property $K^- \subsetneq N$.

Proposition 3. *Let $X \in H^r(M)$ be globally C^r-structurally stable vector field, $r \geq 1$. Then $W_X^+ \cap W_X^- = \varnothing$.*

Proof. In [3] it was proved that X has countably many stable and unstable separatrices of saddles at infinity. By Corollary 1 $X \in H^r_{K-S}(M)$, so the union $W_X^\pm$ of all stable (unstable) separatrices of fixed saddles and saddles at infinity is also countable. Suppose that $W_X^+ \cap W_X^- \neq \varnothing$. We choose flowboxes F_1, F_2 such that $\operatorname{cl} F_1 \subset \operatorname{int} F_2$, $W_X^+ \cap W_X^- \cap S_1^\pm \neq \varnothing$, where S_i^+ is the entrance set of F_i, S_i^- is the exit set of F_i and $\rho_H(F_1, F_2) < \varepsilon$ (ρ is a Hausdorff metric). As W_X^+, X_X^- are countable we may assume that edges of F_2 are not segments of trajectories in $W_X^+ \cup W_X^-$. For every $O_X^+(x)$ in W_X^+ and crossing S_2^- there is the last point of intersection. Analogously, there is the first point of intersection for every O_X^- in W_X^- and crossing S_2^+. We denote these sets, respectively, by A and B. Let V be the set of homeomorphisms of M satisfying $\sup_{x \in F_1} \rho(h(x), \mathrm{id}_M(x)) < \varepsilon$. Then V is a neighborhood of id_M in a compact-open topology. By assumption there is a neighborhood U of X in $H^r(M)$ corresponding V and homeomorphism $h_Y \in \mathsf{V}$ conjugating X with Y. Hence $W_Y^+ \cap W_Y^- \cap F_2 \neq \varnothing$ for $Y \in \mathsf{U}$. Let $Y(t) = X + \varepsilon t Z$, where $\varepsilon > 0$, $t \in [0, 1]$. Z is C^∞ vector field perpendicular to X in $\operatorname{int} F_2$ and $Z(x) = 0$ for $x \notin F_2$. For a sufficiently small $\varepsilon > 0$, $Y(t) \in \mathsf{U}$ for all $0 \leq t \leq 1$. As $Y(t)(x) = X(x)$ for $x \notin F_2$ thus $O_{Y(t)}^+(a) = O_X^+(a)$ for $a \in A$, and $O_{Y(t)}^-(b) = O_X^-(b)$ for $b \in B$ which implies $O_{Y(t)}^+(a) \in W_{Y(t)}^+$, $O_{Y(t)}^-(b) \in W_{Y(t)}^-$. Moreover, other stable and unstable separatrices of saddles at infinity are the same for X and $Y(t)$, $t \in [0,1]$. For every $b \in B$ there is a countable set $I_b \subset [0,1]$ such that $O^+_{Y(t)}(B) \cap A \neq \emptyset$. Let $I = \bigcup_{b \in B} I_b$. Thus for $t \in [0,1] - I$, $O^+_{Y(t)} \cap A = \emptyset$ and $W^+_{Y(t)} \cap W^-_{Y(t)} \cap F_2 = \emptyset$. So X cannot be globally C^r structurally stable since X and $Y(t)$ are not conjugated on F_1 and F_2.

To prove the next proposition we need two lemmas.

Lemma 1. *Let S_1, S_2 be compact transverse sections to $X \in H^r_{K-S}(M)$, $r \geq 1$, $P_X : D \to S_2$ be Poincaré map, where $D \subset S_1$. If (a,b) is a component of D then $O^+_X(a)$, $O^+_X(b) \in W^+_X$.*

Proof. Let $x \in D$. Then $O^+_X(x) \cap S_2 \neq \emptyset$ and there is a neighborhood U of x in S_1 such that $O^+_X(y) \cap S_2 \neq \emptyset$ for $y \in U$. Thus D is the countable union of open and connected subsets of S_1. Let (a,b) or $(a,b]$ be such a component of D. We will show that $O^+_X(a) \in W^+_X$.

(i) Assume first that $\omega(O^+_X(a)) \neq \emptyset$. It is clear that if $\omega(O^+_X(a))$ contains a point $p \in \mathrm{Per}(X)$ then p is a fixed saddle. So either $\{p\} \subset \omega(O^+_X(a))$ of $\{p\} = \omega(O^+_X(a))$ which implies that $O^+_X(a) \in W^+_X$. If $\{p\} \subsetneq \omega(O^+_X(a))$ then $\omega(O^+_X(a))$ contains stable and unstable separatrices of fixed saddle p. By S' we denote a transverse section to X at y belonging to the unstable separatrix of p. For each $x \in (a,b)$ the set $A_x = O_X[x, P_X(x)] \cap S'$ is finite. Let $n_x = \text{card } A_x$. Since P_X is a continuous map there is a number $n_0 \in N \cup \{0\}$ such that $n_x = n_0$ for $x \in (a,b)$. On the other hand the assumption $y \in \omega(O^+_X(a))$ implies that $O^+_X(a)$ crosses S' infinitely many times. Then there exists $x_0 \in (a,b)$ sufficiently close to a such that $n_{x_0} > n_0$. This proves that $\omega(O^+_X(a)) = \{p\}$.

(ii) Let $\omega(O^+_X(a)) = \emptyset$, i.e. $O^+_X(a)$ escapes to infinity. As P_X is a homeomorphism $P_X(a,b) = (c,d) \subset S_2$ and P^{-1}_X cannot be extended to $[c,d]$. Analogously like above, it is possible to prove that α-limit set of $O^-_X(c)$ and $O^-_X(d)$ is either a fixed saddle or is empty. We have to show that $c \in J^+_X(a)$ or $d \in J^+_X(a)$. Let $x_n \in (a,b)$, $x_n \to a$. Then $P_X(x_n) = \phi_X(x_n, t_n) \in (c,d)$ and $P_X(x_n) \to c$ or $P_X(x_n) \to d$. Assume that $P_X(x_n) \to c$. If $\omega(O^+_X(a)) = \emptyset$ and $\alpha(O^-_X(c)) = \emptyset$ then $t_n \to +\infty$ and $O^+_X(a)$, $O^-_X(c)$ form a saddle at infinity. If $\omega(O^+_X(a)) = \emptyset$ and $\alpha(O^-_X(c))$ is a fixed saddle p, then stable separatrix $O_X(y)$ of p is contained in the set of accumulation points of $O_X[x_n, P_X(x_n)]$. By (i) $\alpha(O^-_X(y))$ is empty or it is a fixed saddle. The last case is impossible since $X \in H^r_{K-S}(M)$. Thus $y \in J^+_X(a)$ and $O^+_X(a)$, $O^-_X(y)$ form a saddle at infinity. By the same arguments one can prove that $O^+_X(b) \in W^+_X$.

Lemma 2. *Let $X \in H^r_{K-S}(M)$, $r \geq 1$, $O_X(a) \in W^+_X$, $O_X(b) \in W^-_X$ and $O_X(a) \subset \omega(O^+_X(b))$. Then for each neighborhood U of X in $H^r(M)$ there is a vector field $Y \in \mathsf{U}$ such that $W^+_Y \cap W^-_Y \neq \emptyset$.*

Proof. We choose a neighborhood U of $X \in H^r(M)$ and a flowbox F such that $b \in \operatorname{int} F$. Let $S^{\pm}$ be, respectively, the entrance set and the exit set of F, Z be a vector field perpendicular to X at $\operatorname{int} F$ and $Z(x) = 0$ for $x \notin F$. Then for sufficiently small $\varepsilon > 0$ $Y(t) = X + \varepsilon t Z \in \mathsf{U}$ for all $t \in [0, 1]$. By u_1, u_2 we denote the local coordinates defined on an open set $V \supset F$. We may assume that transverse sections S^+, S^- are parallel to axis u_2. Let a' be the first common point of $O^-_X(a)$ and S^+, I be a closed neighborhood of b' in S^+ such that $I \subset \operatorname{int} S^+$. We define $\eta = \inf_{x \in I} |u_2(x) - u_2(P_{Y(1)}(x))|$, where $P_{Y(1)}\colon S^+ \to S^-$ is a Poincaré map of vector field $Y(1)$. Since I is compact $\eta > 0$. Let a_0 denote the last common point of $O_X(a)$ with S^-, b_0 the first common point of $O_X(b)$ with S^+. Then $O^+_{Y(t)}(a_0) = O^+_X(a_0)$, $O^-_{Y(t)}(b_0) = O^-_X(b_0)$ and $O^+_{Y(t)}(a_0) \in W^+_{Y(t)}$, $O^-_{Y(t)}(b_0) \in W^-_{Y(t)}$. Let a_m be the mth intersection point of $O^-_X(a_0)$ with S^-, b_n be the nth intersection point of $O^+_X(b_0)$ with S^+, $m \geq 0$, $n > 0$. Since $O_X(a) \subset \omega(O^+_X(b))$ then there are a_m, b_n satisfying $|u_2(b_n) - u_2(a_m)| < \eta$.

(i) If $u_2(b_n) < u_2(a_m)$ we choose Z directed as axis u_2. By $a_m(t)$, $b_n(t)$ we denote functions assigned, respectively, the mth intersection of $O^-_{Y(t)}(a_0)$ with S^- and nth intersection of $O^+_{Y(t)}(b_0)$ with S^+ for $t \in [0, 1]$. It is clear that $a_m(t)$, $b_n(t)$ are continuous functions defined for sufficiently small t. Moreover, $u_2(a_m(t))$ decreases while $u_2(b_n(t))$ increases for $t \in [0, 1]$ since M is an orientable surface. There are the following cases: either $a_m(t)$ and $b_n(t)$ are defined for all $t \in [0, 1]$ or at least one of them is not defined on the whole $[0, 1]$. In the first case there is $t_0 \in (0, 1)$ such that $u_2(b_n(t_0)) - u_2(a_m(t_0)) = 0$ and consequently $O_{Y(t_0)}(b_n(t_0)) = O_{Y(t_0)}(a_m(t_0)) \in W^+_{Y(t_0)} \cap W^-_{Y(t_0)}$. If $b_n(t)$ is not defined for all $t \in [0, 1]$ then $b_n(t_0)$ belongs to the boundary of some component of D, where $D \subset S^-$ and $P_{Y(t_0)}\colon D^- \to S^+$ is a Poincaré map. By Lemma 1 $O^+_{Y(t_0)}(b_n(t_0)) \in W^+_{Y(t_0)}$, so $O_{Y(t_0)}(b_n(t_o)) \in W^+_{Y(t_0)} \cap W^-_{Y(t_0)}$. The proof is analogous if $a_m(t)$ is not defined for all $t \in [0, 1]$.

(ii) Suppose that $u_2(b_n) > u_2(a_m)$. Then we have to consider a vector field Z directed opposite to axis u_2. Now $u_2(a_m(t))$ increases, $u_2(b_n(t))$ decreases for $t \in [0, 1]$ but the final arguments are the same as in (i).

The next lemma is proved in [3].

Lemma 3. *Let* $O^+_X(x)$ *be an oscillating semitrajectory of vector field* $X \in H^r_{K-S}(M)$, $r \geq 1$. *Then* $\omega(O^+_X(x))$ *contains a saddle at infinity.*

Proposition 4. *A globally* C^r $(r \geq 1)$ *structurally stable vector field* $X \in H^r(M)$ *have no oscillating trajectories.*

Proof. Suppose that $O_X(x)$ is an oscillating trajectory. We may assume that $O^+_X(x)$ oscillates. By Lemma 3 $\omega(O^+_X(x))$ contains a saddle at infinity, i.e. there are $O^+_X(a) \in W^+_X$, $O^-_X(b) \in W^-_X$ and $b \in J^+_X(a)$. Then either (i) $\omega(O^+_X(b)) = \varnothing$

or (ii) $\omega(O_X^+(b))$ is a compact set or (iii) $\omega(O_X^+(b))$ is noncompact subset of M. Since $O_X(b) \subset \omega(O_X^+(x)) \subset \Omega(X) = \{y \in M\colon y \in J_X^+(y)\}$, $b \in J_X^+(b)$. If $\omega(O_X^+(b)) = \varnothing$ then $O_X^+(b)$ and $O_X^-(b)$ form a saddle at infinity and $O_X(b) \in W_X^+ \cap W_X^-$. This contradicts Proposition 3. Assume that $\omega(O_X^+(b))$ is a nonempty compact set. Thus $\omega(O_X^+(b))$ contains a minimal set or it is such one. By Proposition 2 any minimal set is trivial so it is a fixed saddle. Suppose that $\omega(O_X^+(b))$ is a fixed saddle then $O_X(b) \in W_X^+ \cap W_X^-$. In the other case $\omega(O_X^+(b))$ contains a stable separatrix $O_X^+(c)$ of a fixed saddle and $O_X^-(b)$, $O_X^+(c)$ satisfy assumptions of Lemma 2. Then for any neighborhood U of X in $H^r(M)$ there is a vector field $Y \in \mathsf{U}$ satisfying $W_Y^+ \cap W_Y^- \neq \varnothing$. Both cases are impossible by Proposition 3. Let $\omega(O_X^+(b))$ be noncompact subset of M, i.e. $O_X^+(b)$ oscillates. Lemma 3 implies that $\omega(O_X^+(b))$ contains a saddle at infinity $O_X^+(c)$ and $O_X^-(d)$. Applying again Lemma 2 we obtain a contradiction with Proposition 3. This proves Proposition 4.

Proposition 5. *Let $X \in H^r(M)$ be globally C^r structurally stable, $r \geq 1$. Then* $\Omega(X) = \mathrm{Per}(X)$.

Proof. By Corollary 1 and Propositions 3 and 4 $X \in H^r_{K-S}(M)$, X has no oscillating trajectories and $W_X^+ \cap W_X^- = \varnothing$. Suppose that there is $x \in \Omega(X) - \mathrm{Per}(X)$. Then $\omega(O_X^+(x)) = \varnothing$ or $\omega(O_X^+(x))$ is a compact set. In the second case $\omega(O_X^+(x))$ is a fixed saddle and its unstable separatrix $O_X^+(a)$ escapes to infinity. Analogous possibilities are for $\alpha(O_X^-(x))$. Thus we have the following cases:

(i) $\alpha(O_X^-(x)) = \varnothing$, $\omega(O_X^+(x)) = \varnothing$, $x \in J_X^+(x)$, so $O_X(x) \in W_X^+ \cap W_X^-$.

(ii) $\alpha(O_X^-(x)) = \varnothing$, $\omega(O_X^+(x))$ is a fixed saddle and its unstable separatrix $O_X^+(a)$ escapes to infinity. Thus $a \in J_X^+(x)$, $O_X^-(x)$ and $O_X^+(a)$ form a saddle at infinity and $O_X(a) \in W_X^+ \cap W_X^-$.

(iii) $\alpha(O_X^-(x)) = \varnothing$, $\omega(O_X^+(x))$ contains a fixed saddle, its stable separatrix $O_X(a)$ and unstable separatrix $O_X(b)$ such that $\alpha(O_X^-(a)) = \varnothing = \omega(O_X^+(b))$. Then $O_X^-(x)$ and $O_X^+(a)$ satisfy assumption of Lemma 2 and consequently X is conjugated with a vector field Y satisfying $W_Y^+ \cap W_Y^- \neq \varnothing$. Thus also $W_X^+ \cap W_X^- \neq \varnothing$ which contradicts our assumptions. The other three cases with $\omega(O_X^+(x)) = \varnothing$ are symmetric to (i)–(iii).

The next lemma is proved in [2].

Lemma 4. *Let F be a flowbox of $X \in H^r(M)$, $p \in \operatorname{int} S^+$, U be a neighborhood of X in $H^r(M)$, $r \geq 1$. Then there exist a neighborhood S_1^+ of p in S^+ and a flowbox $F_1 \subset F$ with entrance set S_1^+ and corresponding exit set $S_1^- \subset S^-$ such that for any pair of points $q^\pm \in S_1^\pm$ there is a vector field Y satisfying*:

(a) $Y \in \mathsf{U}$,

(b) $Y(x) = X(x)$ *for* $x \notin F$,

(c) $q^- \in O_Y^+(q^+)$ *and* $O_Y[q^+, q^-] \subset F$.

Proposition 6. *For a globally C^r $(r \geq 1)$ structurally stable vector field $X \in H^r(M)$ $\operatorname{cl} W_X^+ \cap \operatorname{cl} W_X^- \subset \operatorname{Per}(X)$.*

Proof. Suppose that $x \in \operatorname{cl} W_X^+ \cap \operatorname{cl} W_X^- - \operatorname{Per}(X)$. We choose a flowbox F such that $x \in \operatorname{int} F$. Since $x \notin \operatorname{Per}(X)$ then by Proposition 5 $x \notin \Omega(X)$. Thus we may assume that $O_X^+(S^-) \cap F = \emptyset$, $O_X^-(S^+) \cap F = \emptyset$, where S^+, S^- are, respectively, the entrance set and the exit set of F. Let $p = O_X^-(x) \cap S^+$, $q = O_X^+(x) \cap S^-$. Since $x \in \operatorname{cl} W_X^+ \cap \operatorname{cl} W_X^-$ there are points (p_n), (q_n) such that $p_n \in S^+$, $q_n \in S^-$, $O_X^-(p_n) \in W_X^-$, $O_X^+(q_n) \in W_X^+$, $p_n \to p$, $q_n \to q$. Let U be a neighborhood of X in $H^r(M)$. By Lemma 4 there are transverse sections $S_1^+ \subset S^+$, $S_1^- \subset S^-$, points $p_n \in S_1^+$, $q_n \in S_1^-$ and a vector field $Y \in \mathsf{U}$ such that $q_n \in O_Y^+(p_n)$. Thus $O_Y(p_n) = O_Y(q_n) \in W_Y^- \cap W_Y^+$ and X is not globally structurally stable by Proposition 3.

Propositions 1, 2, 4, and 6 imply the following:

Theorem. *Let $X \in H^r(M)$ be globally C^r $(r \geq 1)$ structurally stable. Then:*
(a) *every trajectory in* $\operatorname{Per}(X)$ *is hyperbolic,*
(b) X *has no nontrivial minimal sets and no oscillating trajectories,*
(c) $\operatorname{cl} W_X^+ \cap \operatorname{cl} W_X^- \subset \operatorname{Per}(X)$.

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