A NOTE ON EQUIVARIANT ETA INVARIANTS

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ABSTRACT. We prove the regulairty of equivariant eta functions near the origin. We also propose an equivariant version of the Cheeger–Chou index theorem on spaces with conelike singularities.

0. INTRODUCTION

Let M be an odd-dimensional compact Riemannian spin manifold with a fixed spin structure, D the Dirac operator on M. The η function associated to D is defined by [1]

$$(0.1) \qquad \eta(s,D) = \sum_{\lambda \neq 0} (\operatorname{sign} \lambda) \frac{\dim \Gamma(E_\lambda)}{|\lambda|^s},$$

where λ runs over the nonzero eigenvalues of D and $\Gamma(E_\lambda)$ is the eigenspace of λ.

If $T \colon M \to M$ is an isometry preserving the orientation and spin structure and $\widetilde{dT} D = D \widetilde{dT}$, where $\widetilde{dT} \colon \Gamma(S(M)) \to \Gamma(S(M))$ is the lift of $dT \colon \Gamma(TM) \to \Gamma(TM)$ to the spinors, then one can also define the equivariant η function [1, 5] by

$$(0.2) \qquad \eta_T(s,D) = \sum_{\lambda \neq 0} (\operatorname{sign} \lambda) \frac{\operatorname{Tr} \widetilde{dT}|_{\Gamma(E_\lambda)}}{|\lambda|^s}.$$

In the first two sections we prove some basic properties of equivariant η functions and in §3, we point out that a slight modification of Bismut–Cheeger [3] yields an equivariant index theorem for spaces with conelike singularities.

Related work has also been done by J.-M. Bismut and J. Cheeger (unpublished).

1. Equivariant η functions

It is clear that (0.1) and (0.2) are only defined when $\mathrm{Re}(s)$ is large enough. Then by analytic continuation to the whole complex plane, we obtain the meromorphic functions $\eta(s, D)$ and $\eta_T(s, D)$ on $\mathbf{C}$. Of particular interest is their regularity at $s = 0$.

Theorem 1.1. *For* $\mathrm{Re}(s)$ *large enough,*

$$\eta_T(s, D) = \frac{1}{\Gamma((s+1)/2)} \int_0^\infty t^{(s-1)/2} \, \mathrm{tr}(\widetilde{dT} D \exp(-tD^2)) \, dt$$

$$(1.2) \qquad = \frac{1}{\Gamma((s+1)/2)} \int_0^\infty t^{(s-1)/2} \, dt \int_M \mathrm{tr}(\widetilde{dT}_x D \exp(-tD^2)(x, Tx)) \, dx,$$

where dx *is the volume element associated to the metric, and* $D \exp(-tD^2)(x, y)$ *is the kernel of* $D \exp(-tD^2)$.

Proof. For any $\lambda \neq 0$, let $\varphi_{\lambda_1}, \dots, \varphi_{\lambda_{i(\lambda)}}$ be an orthonormal basis of $\Gamma(E_\lambda)$, then it is standard that

$$(1.3) \qquad D \exp(-tD^2)(x, y) = \sum_{\lambda \neq 0} \lambda e^{-t\lambda^2} \sum_{k=1}^{i(\lambda)} \varphi_{\lambda_k}(x) \langle \varphi_{\lambda_k}(y), \cdot \rangle.$$

Let $\{e_i(x)\}$ be an orthonormal basis of S_x, then

$$\widetilde{dT}_x D \exp(-tD^2)(x, Tx) e_i(Tx) = \sum_\lambda \lambda e^{-t\lambda^2} \sum_k \widetilde{dT} \varphi_{\lambda_k}(x) \langle \varphi_{\lambda_k}(Tx), e_i(Tx) \rangle.$$

So

$$\mathrm{tr}\, \widetilde{dT} D \exp(-tD^2)(x, Tx)$$
$$= \sum_\lambda \lambda e^{-t\lambda^2} \sum_k \sum_i \langle \widetilde{dT} \varphi_{\lambda_k}(x), e_i(Tx) \rangle \langle \varphi_{\lambda_k}(Tx), e_i(Tx) \rangle$$
$$= \sum_\lambda \lambda e^{-t\lambda^2} \sum_k \langle \widetilde{dT} \varphi_{\lambda_k}(x), \varphi_{\lambda_k}(Tx) \rangle,$$

and

$$\int_M \mathrm{tr}\, \widetilde{dT} D \exp(-tD^2)(x, Tx) \, dx = \sum_\lambda \lambda e^{-t\lambda^2} \int_M \sum_k \langle \widetilde{dT} \varphi_{\lambda_k}(x), \varphi_{\lambda_k}(Tx) \rangle \, dx$$
$$= \sum_\lambda \lambda e^{-t\lambda^2} \, \mathrm{tr}(\widetilde{dT}|_{\Gamma(E_\lambda)}).$$

Thus

$$\frac{1}{\Gamma((s+1)/2)} \int_0^\infty t^{(s-1)/2} \operatorname{tr}(\widetilde{dT} D \exp(-tD^2))\, dt$$

$$= \frac{1}{\Gamma((s+1)/2)} \int_0^\infty t^{(s-1)/2} \sum_\lambda \lambda e^{-t\lambda^2} \operatorname{tr} \widetilde{dT}|_{\Gamma(E_\lambda)}\, dt$$

$$= \sum_{\lambda \neq 0} (\operatorname{sign} \lambda) \frac{\operatorname{tr} \widetilde{dT}|_{\Gamma(E_\lambda)}}{|\lambda|^s}. \quad \square$$

Corollary 1.4. *If T has no fixed points on M, then $\eta_T(s,D)$ is an entire function.*

Proof. We introduce an auxiliary Grassmann variable z as in [4]. Then we have

$$(1.5) \qquad \exp{-t(D^2 - zD)} = \exp(-tD^2) + zt D \exp(-tD^2),$$

so

$$(1.6) \quad \exp(-t(D^2 - zD))(x,y) = \exp(-tD^2)(x,y) + zt D \exp(-tD^2)(x,y).$$

Since T has no fixed points, $d(x,Tx) > \delta$ for some constant $\delta > 0$. By standard results for elliptic operators, there exist positive constants C_i ($i = 1,2,3,4$) such that as $t \to 0^+$,

$$(1.7) \qquad \| \exp(-t(D^2 - zD))(x,Tx)\| \leq \frac{C_1}{t^{n/2}} \exp(-C_2/t),$$

$$(1.8) \qquad \| \exp(-tD^2)(x,Tx)\| \leq \frac{C_3}{t^{n/2}} \exp(-C_4/t)$$

for all $x \in M$. Here we have assumed that z has "norm" $\|z\| = 1$. By (1.6)–(1.8), there are positive constants C_5, C_6 such that as $t \to 0^+$,

$$(1.9) \qquad \|D \exp(-tD^2)(x,Tx)\| \leq \frac{C_5}{t^{n/2+1}} \exp(-C_6/t).$$

Thus, $\forall s \in \mathbf{C}$

$$\lim_{t \to 0} |t^{(s-1)/2} \operatorname{tr}(\widetilde{dT} D \exp(-tD^2))|$$

$$= \lim_{t \to 0} |t^{(s-1)/2}| \cdot \left| \int_M \operatorname{tr} \widetilde{dT} D \exp(-tD^2)(x,Tx)\, dx \right|$$

$$\leq \lim_{t \to 0} |t^{(s-1)/2}| \cdot C_7 \frac{C_5}{t^{n/2+1}} \exp(-C_6/t)$$

$$(1.10) \qquad = 0,$$

where C_7 is a suitable positive constant.

Thus $\int_0^\infty t^{(s-1)/2} \operatorname{tr}(\widetilde{dT} D \exp(-tD^2))\, dt$ is an entire function. On the other hand, $1/\Gamma((s+1)/2)$ is also an entire function, (1.4) follows from Theorem 1.1. $\square$

Remark 1.11. In fact, (1.4) is a known result and can be proved directly without introducing the Grassmann variable z.

2. REGULARITY NEAR $s = 0$

In this section, we prove the following theorem:

Theorem 2.1. $\eta_T(s,D)$ *is holomorphic on* $\mathrm{Re}(s) > -2$.

First note that it is sufficient to show that

$$(2.2) \qquad \left(\frac{1}{t}\right)^{1/2} |\operatorname{tr} \widetilde{dT} D \exp(-tD^2)| \le C, \qquad \text{as } t \to 0^+$$

for some constant $C > 0$. Because then the integration in (1.2) will be convergent absolutely and uniformly on the compact subsets in $\mathrm{Re}(s) > -2$, which implies the holomorphic property.

Now we begin to prove (2.2).

Since T is an isometry, the fixed point set F of T consists of components $F_1, \ldots, F_k$, each of even codimension. If U is an open neighborhood of F, then by the argument in the proof of (1.4), it is easy to see that

$$(2.3) \qquad \lim_{t \to 0} \left(\frac{1}{t}\right)^{1/2} \int_{M \setminus U} \operatorname{tr} \widetilde{dT} D \exp(-tD^2)(x, Tx)\, dx = 0,$$

so

$$(2.4) \qquad \lim_{t \to 0} \left(\frac{1}{t}\right)^{1/2} \int_M \operatorname{tr} \widetilde{dT} D \exp(-tD^2)(x, Tx)\, dx$$
$$= \lim_{t \to 0} \left(\frac{1}{t}\right)^{1/2} \int_U \operatorname{tr} \widetilde{dT} D \exp(-tD^2)(x, Tx)\, dx.$$

Thus we meet a local problem, and the situation is similar to what was considered in [8]. As in [8], we may assume $k = 1$ and $\operatorname{codim} F = 2n'$. Denote by $N(F)$ the normal bundle to F. We need only to prove that

$$(2.5) \qquad \lim_{t \to 0} \left(\frac{1}{t}\right)^{1/2} \left| \int_F \int_{N_\xi(\varepsilon)} \operatorname{tr} \widetilde{dT} D \exp(-tD^2)(x, dTx)\, dN_\xi\, d\xi \right| \le C,$$

for some constant $C > 0$. Here $N_\xi(\varepsilon) = \{v \in N_\xi(F) | \|v\| < \varepsilon\}$, is a sufficiently small neighborhood of the origin in the normal space $N_\xi(F)$, and we use the local trivialization of $\exp N_x(\varepsilon)$ to $N_x(\varepsilon)$.

We choose an orthonormal basis and associated coordinates as in [8]. And we define as in [9, 8],

$$(2.6) \qquad \chi(x^\alpha D_x^\beta e^\gamma) = |\beta| - |\alpha| + |\gamma|,$$

for $\alpha, \beta \in Z^n$, $\gamma \in (Z_2)^n$. We also define $\chi(z) = 1$. Then we have

$$(2.7) \qquad \chi(z x^\alpha D_x^\beta e^\gamma) = 1 + |\beta| - |\alpha| + |\gamma|.$$

Now as in [6], set

$$(2.8) \qquad h(x) = 1 + \frac{1}{2} z \sum_{i=1}^{n} x_i e_i .$$

Then it is trivial to verify that

$$
\begin{aligned}
h e_i h^{-1} &= e_i + x_i z , \\
(2.9) \qquad &= e_i + (\chi = 0) ,
\end{aligned}
$$

and

$$(2.10) \qquad h(D^2 - zD)h^{-1} = D^2 + zu ,$$

where $\chi(u) \leq 0$ and u contains no z. Also write (2.10) as

$$(2.11) \qquad D^2 - zD = h^{-1}(D^2 + zu)h .$$

Thus,

$$(2.11)' \qquad \exp(-t(D^2 - zD))(x,y) = h^{-1}(x) \exp(-t(D^2 + zu))(x,y)h(y) .$$

By (1.6) and (2.11)',

$$
\begin{aligned}
ztD \exp(-tD^2)(x,y) &= h^{-1}(x) \exp(-t(D^2 + zu))(x,y)h(y) \\
(2.12) \qquad &\quad - \exp(-tD^2)(x,y) .
\end{aligned}
$$

Then

$$
\begin{aligned}
zt\widetilde{dT}D \exp(-tD^2)(x,dTx) &= \widetilde{dT}h^{-1}(x) \exp(-t(D^2 + zu))(x,dTx)h(dTx) \\
(2.13) \qquad &\quad - \widetilde{dT} \exp(-tD^2)(x,dTx) .
\end{aligned}
$$

As in [9] (compare also with [4] or [7]), it is easy to see that

$$(2.14) \quad \exp(-t(D^2 + zu))(x,y) = \frac{e^{-d(x,y)^2/4t}}{(4\pi t)^{n/2}} \left(\sum_{i=0}^{[n/2]+2} (U_i + zV_i)t^i + o(t^{[n/2]+2}) \right)$$

$$(2.15) \qquad \exp(-tD^2)(x,y) = \frac{e^{-d(x,y)^2/4t}}{(4\pi t)^{n/2}} \left(\sum_{i=0}^{[n/2]+2} U_i t^i + o(t^{[n/2]+2}) \right) ,$$

where $\chi(U_i) \leq 2i$, $\chi(V_i) \leq 2(i-1)$ and U_i, V_i contain no z.

By (2.9) and (2.13)–(2.15), we get

$$
\begin{aligned}
&t\widetilde{dT}D \exp(-tD^2)(x,dTx) \\
&= \frac{e^{-d(x,dTx)^2/4t}}{(4\pi t)^{n/2}} \widetilde{dT} \left[\left(\sum_i ((dT - I)x)_i e_i \right) \cdot \left(\sum_{i=0}^{[n/2]+2} U_i t^i + o\left(t^{[n/2]+2}\right) \right) \right. \\
(2.16) \qquad & \\
&\qquad\qquad\qquad \left. + \left(\sum_{i=0}^{[n/2]+2} W_i t^i + o\left(t^{[n/2]+2}\right) \right) \right] ,
\end{aligned}
$$

where $\chi(W_i) \leq 2(i-1)$.

Lemma 2.17. *Suppose* $i \le [n/2] + 2$. *If* W *is an odd element and* $\chi(W) \le 2i - 2 + 2n'$, *then*

$$(2.18) \qquad \lim_{t \to 0} \left(\frac{1}{t}\right)^{3/2} \left| \int_{N_\xi(\varepsilon)} \frac{e^{-d(x, dTx)^2/4t}}{(4\pi t)^{n/2}} \operatorname{tr}(W(0;x)) t^i \, dx \right| \le C_1,$$

for some constant $C_1 > 0$; *where in the* $W(y;x)$, y *stands for tangential coordinates and* x *stands for normal coordinates.*

Proof. We can assume that W is a monomial, then it can be written as

$$(2.19) \qquad W = \varphi(0) x_{i_1} \cdots x_{i_k} \cdot e_1 \cdots e_n.$$

Here $e_1 \cdots e_n$ all appear because otherwise, $\operatorname{tr} W = 0$. Also note that we can assume that the x_i's in (2.19) are normal coordinates, for otherwise $\operatorname{tr} W(0;\cdot) = 0$.

(i) If $\chi(W) = 2i - 2 + 2n'$, then $k = n + 2 - 2n' - 2i$. By making the change of variables $x = t^{1/2} b$, we get

$$\lim_{t \to 0} \left| \left(\frac{1}{t}\right)^{3/2} \int_{N_\xi(\varepsilon)} \frac{e^{-d(x, dTx)^2/4t}}{(4\pi t)^{n/2}} \operatorname{tr}(W) t^i \, dx \right|$$

$$\le \lim_{t \to 0} \frac{C_2}{t^{3/2}} \left| \int_{N_\xi(\varepsilon/\sqrt{t})} \frac{e^{-\|(I - dT)b\|^2/4}}{(4\pi t)^{n/2}} t^{n/2 + 1 - n' - i} b_{i_1} \cdots b_{i_k} t^i t^{n'} \, db \right|$$

$$= 0$$

(because k is odd).

(ii) If $\chi(W) < 2i - 2 + 2n'$, then it is clear that

$$\lim_{t \to 0} \left| \left(\frac{1}{t}\right)^{3/2} \int_{N_\xi(\varepsilon)} \frac{e^{-d(x, dTx)^2/4t}}{(4\pi t)^{n/2}} \operatorname{tr}(W) t^i \, dx \right| \le C_3. \quad \square$$

Now consider the terms in (2.16). First note that the simple argument in [8] shows that

$$(2.20) \qquad \widetilde{dT} = \prod_{\alpha = n - 2n' + 1}^{n} e_\alpha \cdot f + g,$$

where $\chi(f) \le 0$ and $\chi(g) \le 2n' - 2$.

On the otherhand, $dT|_{TF} = Id_{TF}$, so

$$(2.21) \qquad \sum_i ((dT - I)x)_i e_i = \sum_{\alpha = n - 2n' + 1}^{n} ((dT - I)x)_\alpha e_\alpha.$$

By (2.21), (2.20),

$$(2.22) \qquad \chi\left(\widetilde{dT} \cdot \sum_i ((dT - I)x)_i e_i \right) \le 2n' - 2.$$

Thus, all the monomial terms in (2.16) satisfy the condition of Lemma 2.17. So we get

$$(2.23) \qquad \lim_{t \to 0} \left(\frac{1}{t}\right)^{1/2} \left| \int_{N_\xi(\varepsilon)} \mathrm{tr}(\widetilde{dT}D \exp(-tD^2)(x, dTx))\, dx \right| < C_4,$$

for some constant $C_4 > 0$. By combining this with (2.5), (2.2), the proof of Theorem 2.1 is completed. $\square$

Remark 2.24. A more careful look at the proof allows us to write (2.2) in a more precise form:

Theorem 2.25. *If* $T: M \to M$ *is an isometry and* $F = \cup F_i$ *is its fixed point set, then there exist smooth functions* φ_i *on* F_i *such that*

$$(2.26) \qquad \mathrm{tr}\, \widetilde{dT}D \exp(-tD^2) = \sum_i \int_{F_i} \varphi_i(x)\, dx \cdot t^{1/2} + o(t^{1/2}), \qquad t \searrow 0.$$

Remark 2.27. For $T = Id$, (2.1) and (2.25) were obtained by Bismut–Freed [4] who improved an observation of Atiyah, Patodi, and Singer [1].

3. AN INDEX THEOREM

Now that we have established the basic properties of equivariant η functions, an equivariant version of the index theorem of Cheeger and Chou (cf. [3]), can be obtained immediately by a slight modification of what was done in §1 of [3], where a new proof of the Cheeger–Chou theorem was given.

Here we just state the theorem and indicate briefly what should be modified. For notation and other details, we just refer to [3, §1].

Let Z denote a smooth connected compact manifold with smooth compact boundary ∂Z. Assume Z has even dimension, is oriented and spin. Define

$$C(\partial Z) = ((0, 1] \times \partial Z) \cup \{\delta\}$$

and

$$Z' = Z \bigcup_{\partial Z} C(\partial Z).$$

Introduce a metric $g^{Z', \varepsilon}$ on Z' as in [3].

Let $T: Z' \to Z'$ be an isometry, which is a product near ∂Z, i.e. it is a trivial extension of the isometry $T|_{\partial Z}: \partial Z \to \partial Z$ to $(0, 1 + \varepsilon] \times \partial Z$ in a tubular neighborhood of ∂Z. Assume further more that T has no fixed points on ∂Z. Let $D^\varepsilon_\pm$ be the Dirac operator on Z' associated to $g^{Z', \varepsilon}$. Assume $\widetilde{dT}D^\varepsilon_\pm = D^\varepsilon_\pm \widetilde{dT}$. Then we can define the Lefschetz number

$$(3.1) \qquad L(T) = \mathrm{tr}\, \widetilde{dT}|_{\ker D^\varepsilon_+} - \mathrm{tr}\, \widetilde{dT}|_{\ker D^\varepsilon_-}.$$

Suppose as in [3] that

$$(3.2) \qquad \ker D^{\partial Z} = 0,$$

then we can state the theorem as follows.

Theorem 3.3. *For ε sufficiently small,*

$$L(T) = \sum_i \int_{F_i} \widehat{A}(TF_i)(Pf(2\sin(\Omega/4\pi + \sqrt{-1}\Theta/2)))(N(F_i)))^{-1}$$

(3.3) $- \frac{1}{2}\eta_T(0, D^{\partial Z}),$

where the F_i are the components of the fixed point set of T in Z, and the integrand is the standard density in Lefschetz fixed point formulas (cf., e.g. [8] or [2]).

The proof is almost the same as what was done in [3, §1]. The first point is to note that here we just use

(3.5) $L(T) = \mathrm{tr}_s\, \widetilde{dT}\exp(-t(D^\varepsilon)^2)$

to replace the corresponding

(3.6) $\mathrm{ind}\, D^\varepsilon_+ = \mathrm{tr}_s(\exp(-t(D^\varepsilon)^2))$

in [3]. Then the proof is almost line by line the same as there. We have only to insert $\widetilde{dT}$ in all kernels, e.g. we write

(3.7) $\widetilde{dT}P^\varepsilon_s((r,x),(r,Tx))$

to replace

(3.8) $P^\varepsilon_s((r,x),(r,x))$

in [3]. The only difference worth mentioning is that the equivariant replacement of (1.44) in [3] can be proved easily by using the fact that T has no fixed points on ∂Z and the proof of Corollary 1.4 in this note.

Details are omitted.

Also note that the index theorem of Atiyah–Patodi–Singer and Donnelly [5] for G-manifolds with boundary can also be deduced from (3.3) by a simple argument, similar to what was done in §1, d) of [3].

Remark 3.9. The condition (3.2) and the assumption that T has no fixed points on the boundary ∂Z are not essential. Indeed, this has been treated by J.-M. Bismut and J. Cheeger (unpublished). Here we will not go into the analytical details.

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