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LIAPANOV'S AND RELATED THEOREMS

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ABSTRACT. Four theorems related to vector-valued measures, are the consequence of a single approach.

1. INTRODUCTION

The theorems proved are Liapanov's theorem on the convexity of vector-valued measures, Krein's theorem on the convexity of moment sets associated with Chebychev systems, the Hobby–Rice and the Phelps–Dye theorems on the existence of extreme annihilators of subspaces of $L_1[0,1]$ and $L_1(X,\mu)$, respectively.

Like A. Pinkus' charming proof of the Hobby–Rice theorem, we use the Borsuk antipodal mapping theorem. Other than that and introductory functional analysis results, the paper is self-contained. The relation of results here to other works and historical references are outlined in a "notes" section at the end.

Notation. Let E be a subspace of a normed space B. We use B^* to represent the continuous linear functionals on B. $S(B) = \{b \in B: \|b\| \leq 1\}$, and $B^\perp = \{L \in B^*: L(e) = 0 \text{ for all } e \in E\}$. An $L \in B^*$ is said to *annihilate* E if $L \in E^\perp$. The real polynomials of degree $\leq n$ are denoted by P_n. The extreme points of a convex set K are written $\operatorname{ext}(K)$. The zero set, support set, and sign of a function f are abbreviated to $Z(f)$, supp f, and sgn f, respectively.

Definition. A function g defined on an appropriate set is *odd* if $g(-x) = -g(x)$.

Definition. Let F be a subspace of B. A norm one functional $L \in B^*$ is a *support functional* for a norm one vector $f \in F$, if $L(f) = 1$. F is a *smooth* subspace if each norm one vector in F has a unique support functional.

Convention. We will assume throughout that (X,μ) represents a finite non-atomic measure space.

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Borsuk antipodal mapping theorem. *A continuous odd mapping from the surface of an $(n+1)$-dimensional sphere into an n-dimensional space, has a zero.*

Smooth spaces, the Hobby–Rice and the Phelps–Dye theorems

Lemma 1. *Suppose*

(i) *E and F are subspaces of a normed space,*
(ii) *E has dimension n,*
(iii) *F is a smooth space of dimension $n+1$.*

Then the support functional for some member of F annihilates E.

Proof. Let $(e_1, \dots, e_n)$ be a basis for E. Since F is smooth each $L \in F^*$ of norm one has a unique Hahn–Banach extension L^*. From the Alaoglu theorem, and the smoothness of F, $L \to (L^*e_1, \dots, L^*e_n)$ is continuous. Since the mapping is also odd, it has a zero. □

Lemma 2. *A subspace $F \subset L_1(X, \mu)$ is smooth if and only if $f \in F - \{0\}$ implies $\mu(Z(f)) = 0$.*

Proof. $L \in S(L_\infty(X, \mu))$ represents a supporting functional at f if and only if $L = \operatorname{sgn} f$ on $\operatorname{supp} f$. □

Theorem (Hobby–Rice). *If E is an n-dimensional subspace of $L_1[0,1]$, for some $m \leq n$ there are points $0 = x_0 < x_1 < \cdots < x_n < x_{m+1} = 1$ such that for all $g \in E$, $\sum_{i=1}^{n+1} (-1)^i \int_{x_{i-1}}^{x_i} g\, d\mu = 0$.*

Proof. Let $F = P_n$ in Lemma 1. □

Suppose that (X, μ) contains a nested family of measurable subsets $\{U_i\}_{i \in [0,1]}$ such that $\mu(\{U_i\}) = i$. Put $g(x) = \min\{i : x \in U_i\} = \max\{i : x \notin U_i\}$.

Lemma 3. *$\{p(g(x) : p \in P_n\}$ is a smooth subspace of $L_1(X, \mu)$ of dimension $n+1$.*

Proof. Since $\{x : g(x) = c\} \subset U_{c+1/n} - U_{c-1/n}$, $\mu(\{x : g(x) = c\}) = 0$, and Lemma 2 applies. □

Lemma 4. *$L_1(X, \mu)$ contains smooth subspaces of all finite dimensions.*

Proof. Let $\nu = |\mu|/|\mu(X)|$. It is equivalent to show that $L_1(X, \nu)$ contains smooth subspaces. The measurable sets can be partially ordered by inclusion. Since (X, ν) is positive and nonatomic, a maximal totally ordered family of sets contains a set of measure i for each $0 \leq i \leq 1$. Hence it contains a totally ordered subfamily $\{U_i\}_{i \in [0,1]}$ such that $\mu(\{U_i\}) = i$, and Lemma 3 is applicable. □

Theorem (Phelps–Dye). *For a finite-dimensional subspace E of $L_1(X, \mu)$,* $\operatorname{ext} S(L_\infty(X, \mu)) \cap E^\perp \neq \emptyset$.

Proof. This is immediate from Lemmas 1, 2, and 4. □

THE THEOREMS OF LIAPANOV AND KREIN

Notation. Let $\{g_1, \dots, g_n\} \subset L_1(X, \mu)$. Let $G = \operatorname{span}\{g_1, \dots, g_n\}$. For $h \in L_\infty(X, \mu)$, let $M(h) = (\int_X hg_1\, d\mu, \dots, \int_X hg_n\, d\mu)$.

Liapanov's theorem. *$\{M(s): s \in \operatorname{ext} S(L_\infty(X, \mu))\}$ is compact and convex.*

Comment. From the Alaoglu theorem $\{M(h): h \in S(L_\infty(X, \mu))\}$ is a compact convex set. We want to show that $\{M(h): h \in S(L_\infty(X, \mu))\} \subset \{M(s): s \in \operatorname{ext} S(L_\infty(X, \mu))\}$.

Notation for the proof. Let $h \in S(L_\infty)$. If $\int_X hg\, d\mu = 0$ for all $g \in G$, then an s satisfying the statement of the theorem is given by the Phelps–Dye theorem. Otherwise let G_h be the subspace of G of codimension 1 given by $\{g \in G: \int_X gh\, d\mu = 0\}$. Let g_h be a member of G that has 0 as a best approximation from G_h. From the Hahn–Banach theorem g_h has a support functional $l_h \in G_h^\perp$. Let F be a smooth subspace of $L_1(X, \mu)$ which contains the constants and has dimension $k + 1$ (Lemmas 3 and 4).

Proof of Liapanov's theorem. We seek an $f \in F$ so that $\operatorname{sgn} f \in G_h^\perp$ and $\int_X (\operatorname{sgn} f) g_h\, d\mu = \int_X hg_h\, d\mu$. We use induction on the dimension k of G, and prove the general case first. By the induction hypothesis there is a function s supported on $Z(g_h)$ and having absolute value one there for which $\int_{Z(g_h)} sg\, d\mu = \int_{Z(g_h)} l_h g\, d\mu$ for all $g \in G_h$ Let $t(x) = \{\operatorname{sgn} g_h$ on $\operatorname{supp} g_h$, and s on $Z(g_h)\}$. By multiplying each member of F by t, we may assume that the smooth space F contains a function f_1 for which $\operatorname{sgn} f_1 = t$. In particular $\operatorname{sgn} f_1 \in G_h^\perp$ and is a support functional for g_h. By Lemma 1 there is a nonzero $f_0 \in F$ whose sign annihilates G. Since $G_h^\perp$ has codimension $k - 1$ in L_∞, the functionals obtained by restricting members of $G_h^\perp$ to F has dimension at least equal 2. Furthermore these contain $\operatorname{sgn} f_0$ and $\operatorname{sgn} f_1$. Hence there is a continuous path s_λ from $\operatorname{sgn} f_0$ to $\operatorname{sgn} f_1$ consisting of norm one functionals in F^* whose unique Hahn–Banach extension is in $G_h^\perp$. Since $\int_X s_\lambda g_h\, d\mu$ ranges continuously from 0 to $\|g_h\|_1$ as λ ranges from 0 to 1, there is a λ^* for which $\int_X s_{\lambda^*} g_h\, d\mu = \int_X hg_h\, d\mu$. This s_{λ^*} is realized as the sign of a function in F and we conclude the statement of the theorem.

If the dimension of G is 1, let t be any function of absolute value 1 everywhere that is equal $\operatorname{sgn} g_h$ on $\operatorname{supp} g_h$ and proceed as above. □

Definition. For the following theorem the measure space is Lebesgue measure on the unit interval. However we can still use the definitions of G and $H(h)$ as above. We also use C_n to represent the functions on $[0, 1]$ that have at most n sign changes.

Theorem (Krein). *If G is a smooth subspace of $L_1[0, 1] \cap C_n$ of dimension n or less, then $\{M(s): s \in \operatorname{ext} S(L_\infty \cap C_n)\}$ is a compact convex set.*

Proof. The argument parallels the Liapanov proof. That is, let $h \in S(L_\infty[0, 1])$. We find an s as above such that $M(s) = M(h)$. If $M(h) = 0$, s is given by the

Hobby–Rice theorem. Otherwise define G_h and g_h as in the Liapanov theorem. Let $F = P_m$. Using Lemma 1, there are polynomials p_0 and $p_1 \in P_m$ whose signs annihilate G and G_h, respectively, and $\int(\operatorname{sgn} p_1)g_h = \|g_h\|_1$. Again there is a continuous path of norm one functions $\{p_\lambda : 0 \le \lambda \le 1\}$ in P_m which connect p_0 and p_1 and have sign functions in $G_h^\perp$. Since $\{\int(\operatorname{sgn} p_\lambda)g_h : 0 \le \lambda \le 1\}$ ranges from 0 to $\|g_h\|_1$, there is a λ^* for which $\int(\operatorname{sgn} p_{\lambda^*})g_h = \int hg_h$, and $s = \operatorname{sgn} p_{\lambda^*}$ is the function of the theorem. □

Notes

Lemma 1 is a variation of the Gohberg–Krein theorem and is the only direct use here of the Borsuk theorem. Lemma 1 extends to nonsmooth subspaces as follows:

Corollary. *If E and F are subspaces of a normed space, and have dimension n and $n+1$, respectively, then some support functional for a member of F annihilates E.*

Proof. Let $\|\cdot\|_2$ be an equivalent Hilbert norm on the finite-dimensional space spanned by E and F. Let $\|\cdot\|_k = \|\cdot\| + (1/k)\|\cdot\|_2$. With respect to this new smooth norm there is a functional L_k that attains its norm on F and annihilates E. Letting $k \to \infty$, a cluster point of the associated functionals has the desired properties. □

Corollary (Gohberg–Krein). *If E and F are subspaces of a normed space, and have dimension n and $n+1$, respectively, then some nonzero member of F has zero as a best approximation from E.*

Proof. This follows from the last corollary and the Hahn–Banach theorem. □

The Gohberg–Krein was proved in 1957 with an analogous proof. They observed that the best approximation operator associated with strictly convex spaces is odd and continuous. So if E and F are contained in a strictly convex space the result follows from the Borsuk theorem. The strict convexity hypothesis is removed exactly as in the above corollary. The proof is in G. G. Lorentz' book p. 137, where he applies the result to the theory of n-widths.

In contrast with Lemma 3, not all Banach spaces contain smooth subspaces.

Proposition. *$C(X)$ has no smooth subspaces of dimension > 1.*

Proof. Suppose that $\dim G > 1$. Let $1 \in E$ be a subspace such that $\dim E + 1 = \dim G$. By the Gohberg–Krein theorem some function g in G has 0 as a best approximation from E. Then g has at least two points where it obtains its norm. The associated point evaluation functionals are distinct support functionals. □

R. R. Phelps used the Phelps–Dye theorem to show that no finite-dimensional subspace of $L_1(X, \mu)$ can be a Chebyshev set. Henry Dye showed that the Phelps–Dye theorem was a corollary of the Liapanov theorem.

Allen Pinkus gave a beautiful short proof of the Hobby–Rice theorem. He used the Borsuk theorem directly. He pointed out to us that Hobby and Rice also have a Borsuk argument buried in their proof.

The Hobby–Rice theorem can be extended using Lemma 3 to measure spaces other than $[0,1]$. For example, on the unit square, we get the same result by saying there is an $m \le n$ and m subsquares K_i symmetric about the origin so that $\sum_{i=1}^{m}(-1)^i \int_{K_i - K_{i-1}} g\,d\lambda = 0$ for all $g \in E$. The proof uses a maximal chain of nested squares to build a smooth space F (Lemma 3). For $f \in F$, $Z(f)$ consist of at most n perimeters of such squares. The fact that the Hobby–Rice theorem extends to such parametrizable families of sets was known to A. Pinkus.

Liapanov's Theorem (1940). *The range of a finite nonatomic vector valued measure is a compact convex set.*

The proof was improved by, among others, P. Halmos (1948), D. Blackwell (1951), and Karlin and Studden (1966). In 1966, J. Lindenstrauss found a particularly elegant functional analytic proof.

Proof of the equivalence of this form with that stated in the text. Suppose that $(\mu_1, \dots, \mu_n)$ is the vector valued measure of Liapanov's theorem. Put $\mu = |\mu_1| + \cdots + |\mu_n|$. Since each μ_i is absolutely continuous with respect to μ, each can be written as $\mu_i(K) = \int_K g_i\,d\mu$, for some $g_i \in L_1(X, \mu)$. Hence

$$\{(\mu_1(K), \dots, \mu_n(K)) : K \subset X\} = \left\{\left(\int_K g_1\,d\mu, \dots, \int_K g_n\,d\mu\right) : K \subset X\right\}$$
$$= \left\{\left(\int_X fg_1\,d\mu, \dots, \int_X fg_n\,d\mu\right) : f \text{ an indicator function}\right\}.$$

Now $\{M(h) : 0 \le h \le 1\}$ is a compact (the h's are weak* compact) convex set. So showing that this set is equal $\{M(f) : f \text{ an indicator function}\}$ implies Liapanov's theorem. Furthermore multiplying each function in both sets by 2 and then subtracting 1, shows that the above set equivalence is the same as the statement of the theorem in the text. □

The two forms are equivalent to the following:

Functional Analytic Version. *Let G be a finite-dimensional subspace $L_1(X, \mu)$. For each $h \in S(L_\infty(X, \mu))$, there is an $f \in \operatorname{ext} S(L_\infty(X, \mu))$ such that for all $g \in G$, $\int_X hg\,d\mu = \int_X fg\,d\mu$.*

The equivalence of these forms of the theorem are from Lindenstrauss' paper.

Some applications of the Liapanov theorem are in the works of Hubert Halkin, Blackwell, Dubins and Spannier and Karlin and Studden.

The geometry of the moment set associated with the Krein theorem is extensively examined in Karlin and Studden (pp. 233–241). The result is stated for G an n-dimensional T-system. This assumption requires that

(i) $G \subset C_{n-1}$,

(ii) G is a smooth subspace, and

(iii) members of G are continuous in the open interval. (See the last sentence, p. 233.) However the proof seems to extend to a wider setting.

Open question: characterize the $g_1, \dots, g_n$ for which $\{(\int sg_1, \dots, \int sg_n):$ s has at most m sign changes$\}$ is a compact convex set.

Example. The sets above are not always convex. Let g_1 be the characteristic function of $[0,1] \cup [2,3]$ and g_2 the characteristic function of $[1,2] \cup [3,4]$. Then the above set using $m = 3$ contains $(2,0)$ and $(0,-2)$ but not $(1,-1)$. So it is not convex.

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