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THE SCHUR PRODUCT THEOREM IN THE BLOCK CASE

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ABSTRACT. Let H be a positive semi-definite mn-by-mn Hermitian matrix, partitioned into m^2 n-square blocks H_{ij}, $i, j = 1, \ldots, m$. We denote this by $H = [H_{ij}]$. Consider the function $f: M_n \to M_r$ given by $f(X) = X^k$ (ordinary matrix product) and denote $H_f = [f(H_{ij})]$. We shall show that if H is positive semi-definite then under some restrictions on H_{ij}, H_f is also positive semi-definite. This generalizes familar results for Hadamard and ordinary products.

INTRODUCTION

Let M_n denote the set of n-by-n complex matrices. Let $H = [H_{ij}]$ be a positive semi-definite mn-by-mn Hermitian matrix, partitioned into m^2 n-by-n blocks H_{ij}, $i, j = 1, \ldots, m$. In this paper we are interested in functions $f: M_n \to M_r$ for which $H_f = [f(H_{ij})]$ is positive semi-definite whenever H is positive semi-definite.

Functions of this type have been considered by many authors. The special case $n = r = 1$ leads to consideration of Hadamard products and functions, about which much is known [1, 2, 6]. John de Pillis [5] showed that if H is positive semi-definite and $f(H_{ij})$ is the qth elementary symmetric function of H_{ij} then $H_f \in M_m$ is also positive semi-definite for $1 \le q \le n$; this includes the special case $f(H_{ij}) = \det(H_{ij})$ that had been considered previously by R. C. Thompson [7]. Marvin Marcus and William Watkins [4] showed that if $f(H_{ij}) = \|H_{ij}\|_F^2$ (Frobenius norm of H_{ij}) and H is positive semi-definite then H_f is also positive semi-definite. They also showed that if $H = [H_{ij}]$, $i, j = 1, 2$ is positive semi-definite and $f(H_{ij}) = \operatorname{trace}(H_{ij}^2)$ then H_f is also positive semi-definite. We observed that their second result is not true for 3-by-3 and larger partitions of H.

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Example. Consider the real Hermitian matrix

$$H = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix}$$

whose eigenvalues are $0, 0, 0, 1, 2, 3$. Thus, H is positive semi-definite. Notice that

$$[H_{ij}^2] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

and the eigenvalues of $[H_{ij}^2]$ are $1, 1, 2, 2.414, 0, -0.414$. Now

$$H_f = [\mathrm{trace}(H_{ij}^2)] = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

and $\det(H_f) = -2$.

In view of the fact that H_f is positive semi-definite for $f(H_{ij}) = |H_{ij}|^2$ for $n = 1$, it is plausible to conjecture that if n and m are arbitrary and $f(H_{ij}) = \|H_{ij}\|_{\mathrm{sp}}^2$ then H_f is also positive semi-definite. Unfortunately this is not the case.

Example. Consider the 3-by-3 real Hermitian matrix

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix}$$

We can also partition H as

$$H = \begin{bmatrix} I & B \\ B^* & I \end{bmatrix} \qquad \text{where } B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since $\rho(BB^*) = 1$ H is positive semi-definite. If we consider the function $f(H_{ij}) = \|H_{ij}\|_{\mathrm{sp}}^2$ (spectral norm) then

$$H_f = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and $\det(H_f) = -1$.

In Marcus and William's paper, there is an example to show that if $f(H_{ij}) = H_{ij}^2$ then H positive semi-definite does not imply H_f positive semi-definite even if all H_{ij}'s are Hermitian. However, this function does work for some special types of matrices. One simple preliminary result will be helpful. By $\rho(A)$ we denote the spectral radius of the n-by-n matrix A.

Lemma 1. *Let A be a Hermitian matrix. Then $\rho(A) = \|A\|_{\mathrm{sp}}$.*

Proof. By definition

$$\begin{aligned}\|A\|_{\mathrm{sp}} &= \max\{\sqrt{(\lambda^2)}: \lambda^2 \text{ is an eigenvalue of } A^*A\}\\ &= \max\{\sqrt{(\lambda^2)}: \lambda^2 \text{ is an eigenvalue of } A^2\}\\ &= \max\{|\lambda|: \lambda \text{ is an eigenvalue of } A\}\\ &= \rho(A).\end{aligned}$$

Theorem 2. *If*

$$H = \begin{bmatrix} I & 0 & 0 & \cdots & 0 & B_1 \\ 0 & I & 0 & \cdots & 0 & B_2 \\ 0 & 0 & I & \cdots & 0 & B_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & B_{m-1} \\ B_1^* & B_2^* & B_3^* & \cdots & B_{m-1}^* & I \end{bmatrix}$$

and

$$G = \begin{bmatrix} I & 0 & 0 & \cdots & 0 & A_1 \\ 0 & I & 0 & \cdots & 0 & A_2 \\ 0 & 0 & I & \cdots & 0 & A_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & A_{m-1} \\ A_1^* & A_2^* & A_3^* & \cdots & A_{m-1}^* & I \end{bmatrix}$$

and both are positive semi-definite matrices, then

$$H\Box G = \begin{bmatrix} I & 0 & 0 & \cdots & 0 & B_1A_1 \\ 0 & I & 0 & \cdots & 0 & B_2A_2 \\ 0 & 0 & I & \cdots & 0 & B_3A_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & B_{m-1}A_{m-1} \\ B_1^*A_1^* & B_2^*A_2^* & B_3^*A_3^* & \cdots & B_{m-1}^*A_{m-1}^* & I \end{bmatrix}$$

is also positive semi-definite if $A_iB_i = B_iA_i$ for $i = 1, \ldots, m$.

Proof. Since H and G are positive semi-definite, we have

$$\|B_i\|_{\mathrm{sp}} \le 1, \|A_i\|_{\mathrm{sp}} \le 1 \quad \text{for } i = 1, \ldots, m \tag{3}$$

and also

$$(4)\qquad \left\| \begin{matrix} B_1B_1^* & B_1B_2^* & \cdots & B_1B_{m-1}^* \\ B_2B_1^* & B_2B_2^* & \cdots & B_2B_{m-1}^* \\ \vdots & \vdots & \cdots & \vdots \\ B_{m-1}B_1^* & B_{m-1}B_2^* & \cdots & B_{m-1}B_{m-1}^* \end{matrix} \right\|_{\text{sp}} \leq 1 .$$

We know that $H\square G$ will be positive semi-definite if

$$\rho\left(\left[\begin{bmatrix} B_1A_1 \\ B_2A_2 \\ \vdots \\ B_{m-1}A_{m-1} \end{bmatrix} [B_1^*A_1^*B_2^*A_2^* \cdots B_{m-1}^*A_{m-1}^*]\right]\right) \leq 1 .$$

Now the spectral radius of the matrix within the brackets is

$$\rho\left(\begin{bmatrix} B_1A_1B_1^*A_1^* & B_1A_1B_2^*A_2^* & \cdots & B_1A_1B_{m-1}^*A_{m-1}^* \\ B_2A_2B_1^*A_1^* & B_2A_2B_2^*A_2^* & \cdots & B_2A_2B_{m-1}^*A_{m-1}^* \\ \vdots & \vdots & \cdots & \vdots \\ B_{m-1}A_{m-1}B_1^*A_1^* & B_{m-1}A_{m-1}B_2^*A_2^* & \cdots & B_{m-1}A_{m-1}B_{m-1}^*A_{m-1}^* \end{bmatrix}\right)$$

$$= \rho\left(\begin{bmatrix} A_1B_1B_1^*A_1^* & A_1B_1B_2^*A_2^* & \cdots & A_1B_1B_{m-1}^*A_{m-1}^* \\ A_2B_2B_1^*A_1^* & A_2B_2B_2^*A_2^* & \cdots & A_2B_2B_{m-1}^*A_{m-1}^* \\ \vdots & \vdots & \vdots & \\ A_{m-1}B_{m-1}B_1^*A_1^* & A_{m-1}B_{m-1}B_2^*A_2^* & \cdots & A_{m-1}B_{m-1}B_{m-1}^*A_{m-1}^* \end{bmatrix}\right)$$

[since $A_iB_i = B_iA_i$]

$$= \rho\left(\begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & A_{m-1} \end{bmatrix} \begin{bmatrix} B_1B_1^* & B_1B_2^* & \cdots & B_1B_{m-1}^* \\ B_2B_1^* & B_2B_2^* & \cdots & B_2B_{m-1}^* \\ \vdots & \vdots & \cdots & \vdots \\ B_{m-1}B_1^* & B_{m-1}B_2^* & \cdots & B_{m-1}B_{m-1}^* \end{bmatrix}\right.$$

$$\left. \times \begin{bmatrix} A_1^* & 0 & 0 & \cdots & 0 \\ 0 & A_2^* & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & A_{m-1}^* \end{bmatrix}\right) .$$

By Lemma 1, this is

$$\left\| \begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & A_{m-1} \end{bmatrix} \begin{bmatrix} B_1B_1^* & B_1B_2^* & \cdots & B_1B_{m-1}^* \\ B_2B_1^* & B_2B_2^* & \cdots & B_2B_{m-1}^* \\ \vdots & \vdots & \cdots & \vdots \\ B_{m-1}B_1^* & B_{m-1}B_2^* & \cdots & B_{m-1}B_{m-1}^* \end{bmatrix} \right.$$

$$\left. \times \begin{bmatrix} A_1^* & 0 & 0 & \cdots & 0 \\ 0 & A_2^* & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & A_{m-1}^* \end{bmatrix} \right\|_{\text{sp}},$$

and by the submultiplicative property of the matrix norm it is

$$\leq \left\| \begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & A_{m-1} \end{bmatrix} \right\|_{\text{sp}} \left\| \begin{bmatrix} B_1B_1^* & B_1B_2^* & \cdots & B_1B_{m-1}^* \\ B_2B_1^* & B_2B_2^* & \cdots & B_2B_{m-1}^* \\ \vdots & \vdots & \cdots & \vdots \\ B_{m-1}B_1^* & B_{m-1}B_2^* & \cdots & B_{m-1}B_{m-1}^* \end{bmatrix} \right\|_{\text{sp}}$$

$$\times \left\| \begin{bmatrix} A_1^* & 0 & 0 & \cdots & 0 \\ 0 & A_2^* & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & A_{m-1}^* \end{bmatrix} \right\|_{\text{sp}}$$

≤ 1 by the relationships (3) and (4). Hence $H\Box G$ is positive semi-definite. □

Unfortunately, the □ product of positive definite matrices is not always positive definite if the blocks are greater than 1-by-1. See [4, p. 238] for an example of a positive definite H for which □ product is not positive definite for $n = 2$. The following theorem gives a generalization of the Schur product theorem for special kinds of matrices.

Theorem 5. *Let $H = [H_{ij}]$ be a given positive semi-definite mn-by-mn matrix, assume that each H_{ij} is a normal n-by-n matrix for $i, j = 1, \dots, m$ and assume that the m^2 matrices $\{H_{ij}: 1 \leq i, j \leq m\}$ are a commuting family. If H is positive semi-definite, then so is $[H_{ij}^{\alpha}]$ for all $\alpha = 1, 2, \dots$. If, in addition each H_{ij} is positive semi-definite, then $[H_{ij}^{\alpha}]$ is positive semi-definite for all real $\alpha \geq mn - 2$.*

Proof. Since the m^2 normal matrices $\{H_{ij}: 1 \leq i, j \leq m\}$ are a commuting family, they are simultaneously diagonalizable by a unitary matrix U, i.e.,

$$H_{ij} = U\Lambda_{ij}U^* \qquad \text{for } i, j = 1, \dots, m$$

where each Λ_{ij} is an n-by-n diagonal matrix. Thus we can write

$$H = \begin{bmatrix} U\Lambda_{11}U^* & U\Lambda_{12}U^* & \cdots & U\Lambda_{1m}U^* \\ U\Lambda_{21}U^* & U\Lambda_{22}U^* & \cdots & U\Lambda_{2m}U^* \\ \vdots & \vdots & \cdots & \vdots \\ U\Lambda_{m1}U^* & U\Lambda_{m2}U^* & \cdots & U\Lambda_{mm}U^* \end{bmatrix} = T\Lambda T^*,$$

where

$$T = \begin{bmatrix} U & 0 & 0 & \cdots & 0 \\ 0 & U & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & U \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \cdots & \Lambda_{1m} \\ \Lambda_{21} & \Lambda_{22} & \cdots & \Lambda_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ \Lambda_{m1} & \Lambda_{m2} & \cdots & \Lambda_{mm} \end{bmatrix}.$$

Since H is positive semi-definite and T is unitary, Λ is positive semi-definite. Again since Λ is positive semi-definite the matrix $[\Lambda_{ij}^{\alpha}]$ is also positive semi-definite for $\alpha = 1, 2, \ldots$ by the Schur product theorem. Therefore, any matrix that is unitarily similar to this matrix is also positive semi-definite. Thus for $\alpha = 1, 2, \ldots$

$$T \begin{bmatrix} \Lambda_{11}^{\alpha} & \Lambda_{12}^{\alpha} & \cdots & \Lambda_{1m}^{\alpha} \\ \Lambda_{21}^{\alpha} & \Lambda_{22}^{\alpha} & \cdots & \Lambda_{2m}^{\alpha} \\ \vdots & \vdots & \cdots & \vdots \\ \Lambda_{m1}^{\alpha} & \Lambda_{m2}^{\alpha} & \cdots & \Lambda_{mm}^{\alpha} \end{bmatrix} T^{-1}$$

$$= \begin{bmatrix} U & 0 & 0 & \cdots & 0 \\ 0 & U & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & U \end{bmatrix} \begin{bmatrix} \Lambda_{11}^{\alpha} & \Lambda_{12}^{\alpha} & \cdots & \Lambda_{1m}^{\alpha} \\ \Lambda_{21}^{\alpha} & \Lambda_{22}^{\alpha} & \cdots & \Lambda_{2m}^{\alpha} \\ \vdots & \vdots & \cdots & \vdots \\ \Lambda_{m1}^{\alpha} & \Lambda_{m2}^{\alpha} & \cdots & \Lambda_{mm}^{\alpha} \end{bmatrix} \begin{bmatrix} U^* & 0 & 0 & \cdots & 0 \\ 0 & U^* & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & U^* \end{bmatrix}$$

$$= \begin{bmatrix} U\Lambda_{11}^{\alpha}U^* & U\Lambda_{12}^{\alpha}U^* & \cdots & U\Lambda_{1m}^{\alpha}U^* \\ U\Lambda_{21}^{\alpha}U^* & U\Lambda_{22}^{\alpha}U^* & \cdots & U\Lambda_{2m}^{\alpha}U^* \\ \vdots & \vdots & \cdots & \vdots \\ U\Lambda_{m1}^{\alpha}U^* & U\Lambda_{m2}^{\alpha}U^* & \cdots & U^{\alpha}\Lambda_{mm}^{\alpha}U^* \end{bmatrix} = [H_{ij}^{\alpha}]$$

is positive semi-definite.

If all H_{ij}'s are positive semi-definite then Λ is a real symmetric positive semi-definite matrix with nonnegative entries. We know [1] $\Lambda^{\alpha} = [\Lambda_{ij}^{\alpha}]$ is also positive definite for $\alpha \geq mn - 2$. By the same argument as before it is easy to show that $[H_{ij}^{\alpha}]$ is positive definite for $\alpha \geq mn - 2$. □

The last part of Theorem 5 can be viewed as a generalization of the result on fractional Hadamard powers on positive definite matrices by FitzGerald and Horn [1].

For the function $f(H_{ij}) = \|H_{ij}\|_{\text{sp}}$, there is not hope of proving that H_f is always positive semi-definite whenever H is positive semi-definite, because when $n = 1$, $H_f = |H|$ (entry-wise absolute values). See [3, Chapter 7.5, Problem 6] for an example of a positive definite H for which $|H|$ is not positive definite. However, since the absolute value function does preserve positive definiteness for 2-by-2 matrices we can show that if $f(H_{ij}) = \|H_{ij}\|_{\text{sp}}$, then H_f is positive semi-definite when H is positive semi-definite for $m = 2$, n arbitrary.

Theorem 6. *If $H = [H_{ij}]$ $i, j = 1, 2$, and $f(H_{ij}) = \|H_{ij}\|_{\text{sp}}$, then H_f is positive semi-definite whenever H is positive semi-definite.*

Proof. Since H is positive semi-definite we know that

$$(x^* H_{11} x)(y^* H_{22} y) \geq |x^* H_{12} y|^2 \qquad \text{for all } x, y \in \mathbf{C}^n. \tag{7}$$

By definition

$$\|H_{12}\|_{\text{sp}}^2 = \max_{\|v\|=1} v^* H_{12}^* H_{12} v = z^* H_{12}^* H_{12} z$$

for some $z \in \mathbf{C}^n$. Let $x = H_{12} z$ and $y = z$ then from (7) we have

$$(H_{12}z)^* H_{11} (H_{12}z)(z^* H_{22} z) \geq |(H_{12}z)^* H_{12} z|^2. \tag{8}$$

If H_{12} is the n-by-n zero matrix, then H_f is positive semi-definite. Assume H_{12} is not a zero matrix. Therefore, $(H_{12}z)^*(H_{12}z) \neq 0$. If we divide both sides of the inequality (8) by $(H_{12}z)^*(H_{12}z)$, we have

$$\frac{(H_{12}z)^* H_{11} (H_{12}z)}{(H_{12}z)^*(H_{12}z)} (z^* H_{22} z) \geq \frac{(\|H_{12}\|_{\text{sp}}^2)^2}{(H_{12}z)^*(H_{12}z)}.$$

Now

$$\|H_{11}\|_{\text{sp}} \|H_{22}\|_{\text{sp}} \geq \frac{(H_{12}z)^* H_{11} (H_{12}z)}{(H_{12}z)^*(H_{12}z)} (z^* H_{22} z) \geq \|H_{12}\|_{\text{sp}}^2. \tag{9}$$

This shows that H_f is also positive semi-definite.

The relationship (9) can also be written as

$$\begin{aligned}\|H_{11}\|_{\text{sp}} \|H_{22}\|_{\text{sp}} \geq \rho(H_{11}) \rho(H_{22}) &\geq \frac{(H_{12}z)^* H_{11} (H_{12}z)}{(H_{12}z)^*(H_{12}z)} (z^* H_{22} z) \\ &\geq \|H_{12}\|_{\text{sp}}^2 \geq \{\rho(H_{12})\}^2\end{aligned}$$

which leads us to the following theorem.

Theorem 10. *If $H = [H_{ij}], i, j = 1, 2$, and $f(H_{ij}) = \rho(H_{ij})$, then H_f is positive semi-definite whenever H is positive semi-definite.*

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