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A NONCOMPLETELY REGULAR QUIET QUASI-UNIFORMITY

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ABSTRACT. A complete quiet quasi-uniformity is constructed on A. Mysior's regular but not completely regular space. This answers a question raised by D. Doitchinov at the Prague conference on categorial topology in 1988.

It is well known that the classical theory of completeness for uniform spaces cannot be extended satisfactorily to the class of all quasi-uniform spaces. In [1] D. Doitchinov introduced the class of quiet quasi-uniform spaces and gave a theory of completion for them. His theory of completion is as well behaved as the classical theory of completion for uniform spaces, and his class of quiet quasi-uniform spaces includes all uniform spaces and many of the most interesting quasi-uniform spaces. He developed topological properties of these spaces in [2], where he established that every quiet quasi-uniform space is a regular Hausdorff space and asked whether each quiet space is completely regular. This question was motivated by his previous result that every balanced quasi-metric space is a Tychonoff space [3, Corollary 3]. In this note we construct a quiet transitive quasi-uniformity that is complete in the sense of Doitchinov and is compatible with A. Mysior's noncompletely regular space [6].

The definitions due to Doitchinov needed in this paper are given below; for further information on quasi-uniform spaces see [4].

Definitions. Let $(X,\mathscr{U})$ be a quasi-uniform space and let $\mathscr{F}$ and $\mathscr{G}$ be filters on X. Then $(\mathscr{G},\mathscr{F})\to 0$ provided that for each $U\in\mathscr{U}$, there are $F\in\mathscr{F}$ and $G\in\mathscr{G}$ such that $G\times F\subseteq U$. A T_1-quasi-uniform space $(X,\mathscr{U})$ is *quiet* provided that for each $U\in\mathscr{U}$ there is an entourage $V\in\mathscr{U}$ such that if x' and x'' are points of X and $\mathscr{F}'$ and $\mathscr{F}''$ are filters on X with $(\mathscr{F}'',\mathscr{F}')\to 0$ and if $V(x')\in\mathscr{F}'$ and $V^{-1}(x'')\in\mathscr{F}''$, then $x''\in U(x')$.

A filter $\mathscr{F}$ on X is a *Cauchy filter* provided that there is a filter $\mathscr{G}$ (called a *cofilter* of $\mathscr{F}$) such that $(\mathscr{G},\mathscr{F})\to 0$. A quasi-uniform space in which every Cauchy filter converges is called a *D-complete* space.

We consider the noncompletely regular space of A. Mysior. Its underlying set X is the closed upper half-plane $y\geq 0$ together with one additional point,

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∞. All points above the x-axis are isolated. A basic neighborhood of a point $(x, 0)$ contains $(x, 0)$ and all but finitely many points from the union of two line segments of height 2: the vertical line segment $I_x = \{(x, y): 0 \leq y < 2\}$ and the line segment with slope 1, $J_x = \{(x+y, y): 0 \leq y < 2\}$. The basic neighborhoods of the point ∞ have the form $B_n = \{\infty\} \cup \{(x, y): x > n\}$, where $n = 1, 2, \ldots$.

We define a quasi-uniformity compatible with Mysior's space as follows. For each positive integer n and each finite set K of points above the x-axis let

$$U_{n,K}(z) = \begin{cases} \{z\} & \text{if } z \text{ is above the } x\text{-axis}, \\ B_n & \text{if } z = \infty, \\ I_q \cup J_q - K & \text{if } z = (q, 0). \end{cases}$$

It is easily verified that the collection of all $U_{n,K}$ so defined is a base for a compatible transitive quasi-uniformity $\mathscr{U}$ on X.

The following lemma is a consequence of Theorem 3 [5]. For the sake of completeness, we include its proof.

Lemma 1. *Let $\mathscr{F}$ and $\mathscr{G}$ be filters on X such that $(\mathscr{G}, \mathscr{F}) \to 0$. If $p \in \cap\mathscr{G}$, then $\mathscr{F}$ converges to p.*

Proof. For each $U \in \mathscr{U}$ there are $F \in \mathscr{F}$ and $G \in \mathscr{G}$ with $G \times F \subseteq U$, hence $F \subset U(p)$.

Proposition 1. *The space $(X, \mathscr{U})$ is D-complete.*

Proof. Let $\mathscr{F}$ be a Cauchy filter and suppose that $\mathscr{F}$ does not converge. Let $\mathscr{G}$ be a cofilter for $\mathscr{F}$. There is a positive integer n and a finite set K such that $U_{n,K}(\infty) \notin \mathscr{F}$, and there are $F \in \mathscr{F}$ and $G \in \mathscr{G}$ such that $G \times F \subseteq U_{n,K}$. Note that if $p \in G$ and p lies above the x-axis, then $F \subseteq U_{n,K}(p) = \{p\}$—a contradiction. Moreover, if $\infty \in G$, then $F \subseteq U_{n,K}(\infty)$ which contradicts that $U_{n,K}(\infty) \notin \mathscr{F}$. Therefore G consists only of points on the x-axis. Suppose that $p, q \in G$ and $p \neq q$. Then $F \subseteq U_{n,K}(p) \cap U_{n,K}(q)$, and so there exists a point r lying above the x-axis such that $F = \{r\}$. This contradicts our assumption that $\mathscr{F}$ does not converge. Therefore G has only one point p. By Lemma 1, $\mathscr{F}$ converges to p. This contradiction completes the proof.

Proposition 2. *The quasi-uniformity $\mathscr{U}$ is quiet.*

Proof. Let $U_{n,K} \in \mathscr{U}$ and set $V = U_{n+2,K}$. Let z', z'' be points of X, and let $\mathscr{F}$ and $\mathscr{G}$ be filters on X such that $(\mathscr{G}, \mathscr{F}) \to 0$, $V(z') \in \mathscr{F}$, and $V^{-1}(z'') \in \mathscr{G}$. We show that $(z', z'') \in U_{n,K}$. By Proposition 1, $\mathscr{F}$ converges to some z in X, hence $W(z) \in \mathscr{F}$ for every $W \in \mathscr{U}$. There are $F \in \mathscr{F}$ and $G \in \mathscr{G}$ with $G \times F \subseteq V$. Choose $r \in F \cap V(z')$. Then $G \times \{r\} \subseteq V$, hence $G \subseteq V^{-1}(r)$. Clearly, $V^{-1}(r) \subseteq \{r, \infty, p, q\}$ where p, q lie on the x-axis. As $\mathscr{G}$ is a filter, there exists $w \in \{r, \infty, p, q\}$ such that $w \in \cap\mathscr{G}$. By Lemma 1, $\mathscr{F}$ converges to w. Since X is a Hausdorff space, $w = z$ and $z \in V^{-1}(z'')$.

If $z' = \infty$, then $r \in B_{n+2}$; thus $z \in V^{-1}(r) \subseteq B_n = U_{n,K}(z')$, and so $(z', z'') \in V \circ U_{n,K} = U_{n,K}$.

Therefore we may assume that $z' \neq \infty$. It follows that $z \in V(z')$, for otherwise there is $W \in \mathscr{U}$ such that $W(z) \cap V(z') = \varnothing$, which contradicts the fact that both $W(z)$ and $V(z')$ belong to $\mathscr{F}$. Since $z \in V(z')$ and $z'' \in V(z)$, $(z', z'') \in V^2 = V$.

Doitchinov has established that every uniform space is quiet and that the D-completion of a uniform space is the usual uniform completion [1, Proposition 9]. Therefore every Dieudonné complete topological space admits a D-complete quiet quasi-uniformity. As Mysior's space also admits a D-complete quiet quasi-uniformity, it is natural to ask the following question.

Question. Does every regular T_1 space admit a (D-complete) quiet quasi-uniformity?

Added in proof. H. P. Künzi has shown that a regular T_1 space need not admit a quiet quasi-uniformity.

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