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SOME RIGIDITY PHENOMENA FOR EINSTEIN METRICS

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ABSTRACT. In this note we study the following problem: When must a complete Einstein metric g on an n-manifold with $\mathrm{Ric} = (n-1)\lambda g$ be a constant curvature metric of sectional curvature λ?

1. INTRODUCTION

Some nontrivial examples of complete open manifolds with Ricci-flat metrics have been constructed. For example, the Eguchi-Hanson metric [EH] on the tangent bundle of S^2 is Ricci-flat and locally asymptotically Euclidean; more precisely, the sectional curvature decays at a rate of $\frac{1}{r^6}$. In the Kähler case, M. Anderson, P. Kronheimer, and C. LeBrun [AKL] displayed an infinite dimensional family of complete Ricci-flat Kähler manifolds of complex dimension 2, for which the second homology is infinitely generated. Recently, C. LeBrun [L] observed that $\mathbf{C}^2$ admits a complete Ricci-flat Kähler metric that is not flat; the resulting Riemannian manifold is isometric to the Euclidean Taub-NUT metric discovered by Hawking [Ha]. A natural problem is when a complete Ricci-flat metric on open manifold must be flat. In this note we will observe some rigidity phenomena for Einstein metrics.

Theorem 1. *Let (M^n, g) be a complete open n-manifold with zero Ricci curvature. Suppose that*

$$\nu_M = \lim_{r\to\infty} \frac{\mathrm{vol}(B(p,r))}{\omega_n r^n} > 0.$$

There exists a constant $c_1(n)$ depending only on n, such that if

$$\int_M |R_{ijkl}|^{\frac{n}{2}}\, dg \le c_1(n)\nu_M^{n+1},$$

then (M^n, g) is isometric to $\mathbf{R}^n$, where $B(p,r)$ denotes the geodesic ball of radius r around p, and ω_n denotes the volume of the standard unit ball B^n in $\mathbf{R}^n$.

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M. Anderson [A] proved that for any complete open n-manifold (M^n, g) with zero Ricci curvature and $\nu_M > 0$, if

$$\int_M |R_{ijkl}|^{\frac{n}{2}}\, dg < +\infty,$$

then there is an R_0 such that $M^n \backslash B(p, R_0)$ is diffeomorphic to $(R_0, \infty) \times S^{n-1}/\Gamma$, where S^{n-1}/Γ is a spherical space form. Further if, in addition, M^n is simply connected at infinity, then (M^n, g) is isometric to $\mathbf{R}^n$. R. E. Greene and H. Wu [GW] observed some rigidity phenomena for nonnegatively curved metrics on open manifolds; for example, they proved that if M^n is a complete open nonnegatively curved manifold of dimension $n \geq 3$, which is simply connected at infinity, and if M^n has sectional curvature zero outside some compact subset, then M^n is isometric to $\mathbf{R}^n$. Note that they also assume that M^n is simply connected at infinity. In the compact case, we have

Theorem 2. *Let (M^n, g) be a closed n-manifold with* $\mathrm{Ric} = (n-1)g$. *There is a constant $c_2(n)$ depending only on n, such that if*

$$\int_M |R_{ijkl} - (g_{ik}g_{jl} - g_{il}g_{jk})|^{\frac{n}{2}}\, dg \leq c_2(n) v_M,$$

then g is a metric of constant curvature 1, *where v_M denotes the volume of* M^n.

Recently, L. Z. Gao [G1] proved that there is a constant $\mu = \mu(H, v, n) > 0$, such that for any compact manifold (M^n, g) of dimension $n \geq 4$ with $|\mathrm{Ric}| \leq H$, $\mathrm{vol}(B(x, 1)) \geq v$ for all $x \in M^n$, and

$$\int_{B(x,1)} |R_{ijkl} - (g_{ik}g_{jl} - g_{il}g_{jk})|^{\frac{n}{2}}\, dg < \mu$$

for all $x \in M^n$, then there exists a constant curvature metric with sectional curvature equal to 1. See [G1] for further information.

Very recently, S. Bando kindly pointed out that Theorem 1 is related to some work of his, and a similar version of Theorem 1 was stated in his paper "Bubbling out of Einstein Manifolds", preprint. However, explicit small bounds are given in Theorem 1 and Theorem 2 in this note.

2. The Sobolev inequalities

In the first part of this section we assume that (M^n, g) is a complete open n-manifold with nonnegative Ricci curvature, and for some point p,

$$\nu_M = \lim_{r \to +\infty} \frac{\mathrm{vol}(B(p, r))}{\omega_n r^n} > 0.$$

Notice that ν_M does not depend on the choice of p, and ν_M is therefore the global geometric invariant of M^n.

For each point $x \in M^n$ and $r > 0$, let $\omega_x(r)$ be the Lebesgue measure of the set of all unit vectors $v \in T_xM^n$ such that the geodesic γ_v issuing from

x in the direction v has cut value $\geq r$ i.e., the geodesic segment of γ_v on $[0,r]$ has minimal distance. Notice that $\omega_x(r)$ is nonincreasing in r, and $\omega_x(\infty) = \lim_{r\to\infty}\omega_x(r)$ is the Lebesgue measure of the set of all unit tangent vectors of rays issuing from x. Let $R > r > 0$; it follows from the Bishop-Gromov Comparison Theorem [BC] that

$$\mathrm{vol}(B(x,R)\backslash B(x,r)) \leq \int_r^R \omega_x(r)t^{n-1}\,dt = \omega_x(r)\frac{R^n - r^n}{n}.$$

Thus

$$\omega_x(r) \geq n\frac{\mathrm{vol}(B(x,R)) - \mathrm{vol}(B(x,r))}{R^n - r^n}.$$

Letting $R \to \infty$, we obtain

$$\omega_x(r) \geq \nu_M \,\mathrm{vol}(S^{n-1}), \tag{1}$$

where $\mathrm{vol}(S^{n-1})$ denotes the volume of the unit sphere in $\mathbf{R}^n$. Then letting $r \to \infty$, we show the following.

Proposition 1. *Let M^n be a complete open n-manifold with nonnegative Ricci curvature. Suppose that $\nu_M > 0$. Then*

$$\omega_M(\infty) \geq \nu_M \,\mathrm{vol}(S^{n-1}) > 0,$$

where $\omega_M(\infty) = \inf_{x\in M}\omega_x(\infty)$.

Corollary 1 (Marenich-Toponogov [MT]). *Let M^n be a complete open n-manifold with nonnegative Ricci curvature. Suppose that $\nu_M > 0$. Then there are n linearly independent rays emanating from any $q \in M^n$.*

Throughout the rest of this note, $c_i(n)$'s denote positive constants depending only on n. Let Ω be an arbitrary compact domain on M^n. Let $r > 0$ be such that Ω is contained in $B(x,r)$ for any $x \in \Omega$. It follows from Yau's observation [Y] based on the methods of Croke [C] that

$$c_3(n)\left(\inf_{x\in\Omega}\omega_x(r)\right)^{(n+1)/n}\mathrm{vol}(\Omega)^{(n-1)/n} \leq \mathrm{vol}(\partial\Omega).$$

From (1) we conclude that

$$c_4(n)\nu_M^{(n+1)/n}\,\mathrm{vol}(\Omega)^{(n-1)/n} \leq \mathrm{vol}(\partial\Omega),$$

for any compact domain Ω on M^n. This is the isoperimetric inequality that is in fact equivalent to the following Sobolev inequality:

$$c_4(n)\nu_M^{(n+1)/n}\left(\int_M |f|^{n/(n-1)}\,dg\right)^{(n-1)/n} \leq \int_M |\nabla f|\,dg, \tag{2}$$

for all $f \in C_0^\infty(M)$. The inequality (2) implies that for $n \geq 3$,

$$c_5(n)\nu_M^{2+(2/n)}\left(\int_M |f|^{2n/(n-2)}\,dg\right)^{(n-2)/n} \leq \int_M |\nabla f|^2\,dg, \tag{3}$$

for all $f \in C_0^\infty(M)$.

In the second part of this section we will state the Sobolev inequalities for certain closed n-manifolds. Let (M^n, g) be a closed n-manifold with Ricci curvature $\mathrm{Ric}_M \geq (n-1)g$. Then we have

$$(4)\qquad \left(\int_M |f|^{2n/(n-2)}\, dg\right)^{(n-2)/n} \leq C_6(n) v^{-(2/n)} \left(\int_M |\nabla f|^2\, dg + \int_M |f|^2\, dg\right)$$

for all $f \in C^\infty(M)$, where $C_6(n)$ is a constant depending only on n, and v_M denotes the volume of M^n. This is a well-known result. One can refer to Appendix VI, Theorem 3 in [B] for further information.

3. Proof of Theorem 1

The following argument is quite standard (see [A, DY, G1, G2]). However, in our situation, it is unnecessary to use Moser's iteration to get L^∞ estimates for the curvature tensor. Using the old-fashioned index notation for tensors, we denote the Riemannian metric g by g_{ij}, and the curvature tensor Rm by R_{ijkl}. The Ricci curvature tensor is the contraction $R_{ik} = g^{jl} R_{ijkl}$, where g^{ij} is the inverse of g_{ij}. If the metric g is an Einstein metric, then it was shown in [H] that

$$(5)\qquad \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) = g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj}),$$

where $B_{ijkl} = g^{pr} g^{qs} R_{piqj} R_{rksl}$, and Δ denotes the *rough Laplacian* acting on tensors.

Let (M^n, g) be a complete open n-manifold with zero Ricci curvature. It follows from (5) that

$$(6)\qquad \Delta|Rm| + c_7(n)|Rm|^2 \geq 0,$$

where $|Rm|$ denotes the pointwise norm of the curvature tensor $Rm = R_{ijkl}$. Note that if $n = 3$, g is flat itself. Thus we assume that $n \geq 4$. First multiply (6) by $\eta^2|Rm|^{(n-2)/2}$, where η is a cutoff function of compact support in M^n. Integrating by parts, one obtains

$$(7)\qquad \begin{aligned} c_8(n)\int_M |Rm|(\eta|Rm|^{\frac{n}{4}})^2\, dg &\geq 2\int_M \eta|Rm|^{\frac{n}{4}} \nabla\eta \cdot \nabla|Rm|^{\frac{n}{4}}\, dg \\ &\quad + \frac{2(n-2)}{n}\int_M \eta^2 |\nabla|Rm|^{\frac{n}{4}}|^2\, dg. \end{aligned}$$

Note that for $\varepsilon = \frac{n-2}{n}$,

$$(8)\qquad 2\eta|Rm|^{\frac{n}{4}}\nabla\eta\cdot\nabla|Rm|^{\frac{n}{4}} \geq -\varepsilon\eta^2|\nabla|Rm|^{\frac{n}{4}}|^2 - \frac{1}{\varepsilon}|Rm|^{\frac{n}{2}}|\nabla\eta|^2.$$

Let $\delta_M = \int_M |Rm|^{\frac{n}{2}}\, dg$. Applying the Hölder inequality and the Sobolev inequality (3) to the right side of (7), one obtains

$$(9)\qquad \begin{aligned} &c_8(n)\int_M |Rm|(\eta|Rm|^{\frac{n}{4}})^2\, dg \\ &\leq c_9(n)(\nu_M^{-n-1}\delta_M)^{\frac{2}{n}}\left[\int_M \eta^2|\nabla|Rm|^{\frac{n}{4}}|^2\, dg + \int_M |Rm|^{\frac{n}{2}}|\nabla\eta|^2\, dg\right]. \end{aligned}$$

By (8) and (9), the inequality (7) gives the following

$$\begin{aligned}(10)\qquad &\left[\frac{n-2}{n} - c_9(n)(\nu_M^{-n-1}\delta_M)^{\frac{2}{n}}\right]\int_M \eta^2|\nabla|Rm|^{\frac{n}{4}}|^2\,dg \\ &\le \left[\frac{n}{n-2} + c_9(n)(\nu_M^{-n-1}\delta_M)^{\frac{2}{n}}\right]\int_M |Rm|^{\frac{n}{2}}|\nabla\eta|^2\,dg.\end{aligned}$$

It follows from (10) that there is a constant $c_1(n)$ such that if $\delta_M \le c_1(n)\nu_M^{n+1}$, then

$$(11)\qquad \int_M \eta^2|\nabla|Rm|^{\frac{n}{4}}|^2\,dg \le c_{10}(n)\int_M |Rm|^{\frac{n}{2}}|\nabla\eta|^2\,dg.$$

Now we may choose the cutoff function $\eta = \eta(r)$, where r is the distance function to a fixed point p, satisfying that $\eta(r) = 1$, for $r \le R$; $\eta(r) = 0$, for $r \ge 2R$; and $|\eta'(r)| \le \frac{2}{R}$. Then (11) gives

$$\int_{B(p,R)} |\nabla|Rm|^{\frac{n}{4}}|^2\,dg \le \frac{c_{11}(n)}{R^2}\int_M |Rm|^{\frac{n}{2}}\,dg.$$

Letting $R \to \infty$, one obtains

$$\nabla|Rm| \equiv 0.$$

Thus $|Rm| \equiv$constant. Since δ_M is finite and M^n has infinite volume, we conclude that $Rm \equiv 0$. It follows from [MT] that (M^n, g) is isometric to $\mathbf{R}^n$. This proves Theorem 1.

4. Proof of Theorem 2

The following argument is also quite standard and similar to that given above. Let (M^n, g) be a closed n-manifold with $\mathrm{Ric} = (n-1)g$. Let $\overline{R}_{ijkl}$ denote $R_{ijkl} - (g_{ik}g_{jl} - g_{il}g_{jk})$. It follows from (5) that

$$\Delta\overline{R}_{ijkl} + 2(\overline{B}_{ijkl} - \overline{B}_{ijlk} - \overline{B}_{iljk} + \overline{B}_{ikjl}) = 2(n-1)\overline{R}_{ijkl},$$

from which one obtains

$$(12)\qquad \Delta|\overline{R}m| + c_{12}(n)|\overline{R}m|^2 - 2(n-1)|\overline{R}m| \ge 0,$$

where $|\overline{R}m|$ denotes the pointwise norm of the tensor $\overline{R}m = \overline{R}_{ijkl}$, and $\overline{B}_{ijkl} = g^{pr}g^{qs}\overline{R}_{piqj}\overline{R}_{rksl}$. Note that if $n = 3$, g is a metric of constant curvature. Thus we assume that $n \ge 4$. First multiply (12) by $|\overline{R}m|^{\frac{n-2}{2}}$. Integrating by parts, one obtains

$$\begin{aligned}(13)\qquad &c_{13}(n)\int_M |\overline{R}m||\overline{R}m|^{\frac{n}{2}}\,dg \\ &\ge c_{14}(n)\int_M |\overline{R}m|^{\frac{n}{2}}\,dg + \int_M \left|\nabla|\overline{R}m|^{\frac{n}{4}}\right|^2\,dg.\end{aligned}$$

Let $\delta_M = \int_M |\overline{R}m|^{\frac{n}{2}}\, dg$. Applying the Hölder inequality and the Sobolev inequality (4) to the right side of (13), one obtains

$$
\begin{aligned}
&c_{13}(n)\int_M |\overline{R}m||\overline{R}m|^{\frac{n}{2}}\, dg \\
(14)\qquad &\le c_{15}(n)(v_M^{-1}\delta_M)^{\frac{n}{2}}\left[\int_M |\overline{R}m|^{\frac{n}{2}}\, dg + \int_M \left|\nabla|\overline{R}m|^{\frac{n}{4}}\right|^2 dg\right].
\end{aligned}
$$

From (13) and (14) one concludes that

$$
\begin{aligned}
&\left[1 - c_{15}(n)(v_M^{-1}\delta_M)^{\frac{n}{2}}\right]\int_M \left|\nabla|\overline{R}m|^{\frac{n}{4}}\right|^2 dg \\
(15)\qquad &\le \left[c_{15}(n)(v_m^{-1}\delta_M)^{\frac{n}{2}} - c_{14}(n)\right]\int_M |\overline{R}m|^{\frac{n}{2}}\, dg.
\end{aligned}
$$

It follows from (15) that there is a constant $c_2(n)$ such that if $\delta_M \le c_2(n)v_M$, then $\overline{R}m \equiv 0$. This proves Theorem 2.

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