

MINIMAL SURFACES WITH LOW INDEX IN THE THREE-DIMENSIONAL SPHERE

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ABSTRACT. In the present paper, the author gives a characterization of the Clifford torus among the minimal surfaces of the three-dimensional sphere in terms of its index.

1. INTRODUCTION

Let M be a compact orientable minimal surface in the three-dimensional unit sphere $S^3(1)$. The Jacobi operator of the second variation is given by $L = \Delta + |\sigma|^2 + 2$, where σ is the second fundamental form of M and Δ is the Laplacian of the induced metric. The index of M , $\text{ind}(M)$, is defined as the number of negative eigenvalues of L . We remark that the stability of M is equivalent to $\text{Ind}(M) = 0$.

In [S], J. Simons proved that M cannot be stable ($\text{Ind}(M) \geq 1$) and characterized the totally geodesic immersion as the only one with index one.

In this paper, we get a lower bound of $\text{Ind}(M)$ when M is not totally geodesic and characterize the Clifford torus as the only compact minimal surface whose index is that lower bound. In fact, we prove the following:

Theorem. *Let M be a compact orientable nontotally geodesic minimal surface in $S^3(1)$. Then $\text{ind}(M) \geq 5$, and the equality holds if and only if M is the Clifford torus.*

Remark. The above theorem is also true when M is a complete orientable minimal surface of $S^3(1)$, because in $S^3(1)$ any complete orientable minimal surface with finite index is compact. (See [FC] and [L-R], Theorem 4).

2. PRELIMINARIES

Let $\Phi: M \rightarrow S^3(1)$ be an isometric immersion of a compact orientable minimal surface in $S^3(1)$. We denote by $\langle \cdot, \cdot \rangle$ the metric of \mathbb{R}^4 as well as that

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induced on $S^3(1)$ and M . If σ is the second fundamental form of Φ and Δ the Laplacian of the metric induced on M , then $L = \Delta + |\sigma|^2 + 2$ is the Jacobi operator of the second variation, and its associated quadratic form is given by

$$Q(u, u) = \int_M \{ |\nabla u|^2 - (|\sigma|^2 + 2)u^2 \} dA,$$

where dA is the measure of the metric on M . Then, $\text{ind}(M)$ is the index of Q . If $\lambda_i, i = 1, 2, \dots$, are the eigenvalues of L , and f_i the corresponding eigenfunctions, then

$$(1) \quad \lambda_k \leq Q(u, u)$$

for any function u with $\int_M u^2 dA = 1$ and $\int_M u f_i dA = 0$ for $i = 1, 2, \dots, k - 1$, and the equality in (1) holds if and only if $Lu + \lambda_k u = 0$.

Finally (see [M-R] for details), if \mathbb{D}^4 is the open unit ball in \mathbb{R}^4 , then for each $g \in \mathbb{D}^4$ we define a conformal transformation F_g of $S^3(1)$ by

$$F_g(p) = \frac{p + (\mu\langle p, g \rangle + \lambda)g}{\lambda(1 + \langle p, g \rangle)},$$

where $\lambda = (1 - |g|^2)^{-(1/2)}$, $\mu = (\lambda - 1)|g|^{-2}$.

If $\Phi: M \rightarrow S^3(1)$ is a minimal immersion, then

$$(2) \quad 2A = \int_M |\nabla(F_g \circ \Phi)|^2 dA + 2 \int_M \left(\frac{\langle N, g \rangle}{(1 + \langle \Phi, g \rangle)} \right)^2 dA,$$

where A is the area of M , and N is a unit normal vector field to M .

3. PROOF OF THE THEOREM

For any vector $a \in \mathbb{R}^4$ let $f_a = \langle N, a \rangle$. Then it is easy to see that $Lf_a - 2f_a = 0$. So, -2 is an eigenvalue of L , and the functions on $V = \{f_a/a \in \mathbb{R}^4\}$ are eigenfunctions of -2 .

It is clear that $\dim V \leq 4$. If $\dim V \leq 3$, then there exists a nonzero vector a in \mathbb{R}^4 such that $f_a = 0$. Then a simple computation proves that the Hessian of the function $g = \langle \Phi, a \rangle$ satisfies $\text{Hess } g = -g \langle \cdot, \cdot \rangle$, and so using Obata's theorem [O], we have that M is either isometric to a unit sphere or $g = 0$. In the first case, M is totally geodesic. So g must be zero. But in this case, as $a \neq 0$, $\Phi(M)$ lies in an equator of $S^3(1)$, and again M is totally geodesic. So $\dim V = 4$. Then -2 cannot be the first eigenvalue of L , because the multiplicity of λ_1 is one. So $\text{ind}(M) \geq 1 + \dim V = 5$. Now suppose that $\text{ind}(M) = 5$. Then $\lambda_2 = -2$. Let ρ be the eigenfunction of λ_1 which can be taken positive. Then (see [L-Y]), there exists a conformal transformation F_g of $S^3(1)$ such that

$$\int_M \rho(F_g \circ \Phi) dA = 0.$$

So from (1), we have that the components of $F_g \circ \Phi$ satisfy

$$(3) \quad Q((F_g \circ \Phi)_i, (F_g \circ \Phi)_i) \geq -2 \int_M (F_g \circ \Phi)_i^2 dA$$

for $i = 1, 2, 3, 4$. So from (3) we have

$$\int_M |\nabla(F_g \circ \Phi)|^2 dA \geq \int_M |\sigma|^2 dA.$$

Then using (2), we get

$$(4) \quad 2A \geq \int_M |\sigma|^2 dA,$$

and the equality holds if and only if the equality holds in (3) and $\langle N, g \rangle = 0$.

From Gauss's formula, we have $|\sigma|^2 = 2 - 2K$, where K is the Gauss curvature of M . So using the Gauss-Bonnet theorem, (4) implies that the genus of M is less than or equal to one. If the genus is zero, using a result of Almgren [A], we have that M is totally geodesic, which contradicts the assumptions. So M must be a torus, and in this case the equality in (4) holds. So $\langle N, g \rangle = 0$. If $f = \langle \Phi, g \rangle$, then $\text{Hess } f = -f \langle \cdot, \cdot \rangle$, and using a similar reasoning as above, we get that $g = 0$, and then $F_g \circ \Phi = \Phi$.

Now as the equality in (3) holds, too, we have that the components Φ_i of Φ are eigenfunctions of -2 , which implies that $\Delta \Phi_i = -|\sigma|^2 \Phi_i$. But it is easy to see that $\Delta \Phi_i = -2\Phi_i$. So $(|\sigma|^2 - 2)\Phi_i = 0$ for $i = 1, 2, 3, 4$. Then $|\sigma|^2 = 2$, and using a very well-known result (see [Ch-DoC-K]), we obtain that M is the Clifford torus.

Finally, if M is the Clifford torus, we have that $|\sigma|^2 = 2$ and so $L = \Delta + 4$. If μ_i are the eigenvalues of Δ , then $\lambda_i = \mu_i - 4$. As $\mu_1 = 0$, $\mu_2 = 2$ (with multiplicity 4), and $\mu_3 = 4$, we have that $\text{Ind}(M) = 5$.

Remark. The referee pointed out to me that the same theorem was observed independently by Doris Fischer-Colbrie, who apparently lectured on it but never published it.

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