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A NEW FAMILY OF ENNEPER TYPE MINIMAL SURFACES

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ABSTRACT. An Enneper type surface is a complete immersed minimal surface in $\mathbf{R}^3$ with only one end and finite total curvature. In this paper we construct a family of Enneper type surfaces of genus 1, total curvature $-8(2n+1)\pi, n = 0, 1, 2, \cdots$. We use the Weierstrass $\wp$ elliptic function as a tool and also prove some results about $\wp$ on a square torus.

1. INTRODUCTION

An Enneper type minimal surface is a complete immersed minimal surface with finite total Gauss curvature and only one end; (i.e., conformally it is a closed genus k Riemannian surface with one puncture). The simplest example is Enneper's surface. It has genus 0 and total curvature -4π. There is also a family of genus 0 examples with total curvature $-4n\pi, n = 1, 2, 3, \cdots$. In [2] Chen and Gackstätter constructed genus 1 and genus 2 examples with total curvature -8π and -12π. In [6] Wohlgemuth constructed a family of genus 1 examples with total curvature $-4\pi(2n+1)$ for $n \geq 1$. In this paper we will construct a family of genus 1 examples with total curvature $-8\pi(2n+1)$ for $n \geq 0$. Our main tools are Weierstrass representations for minimal surfaces and the Weierstrass elliptic function $\wp$ associated to a lattice $L = [1, \tau]$ in the complex plane $\mathbf{C}$. The by-products of this study are some properties of the Weierstrass $\wp$ function. Having not seen these properties in publication, we list them as a theorem in this paper.

2. WEIERSTRASS REPRESENTATION

A very important tool used in the construction of minimal surfaces is the Weierstrass representation formula. Here we state one version of it; for details see [4] and [5].

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Proposition 1. *Let $\overline{M}$ be a compact Riemann surface and $M = \overline{M} - \{p_1, \cdots, p_n\}$. Suppose $\overline{g}: \overline{M} \to \mathbf{C} \cup \{\infty\}$ is a meromorphic function and η is a meromorphic 1-form such that whenever $g = \overline{g}|M$ has a pole of order k, then η has a zero of order $2k$ and η has no other zeros on M. Let*

$$\omega_1 = \frac{1}{2}(1 - g^2)\eta, \qquad \omega_2 = \frac{i}{2}(1 + g^2)\eta, \qquad \omega_3 = g\eta.$$

If for any closed curve γ in M,

$$\text{(1)} \qquad \operatorname{Re} \int_\gamma \omega_i = 0, \textit{ for } i = 1, 2, 3,$$

then the surface S, defined by $X: M \to \mathbf{R}^3$, is a regular minimal surface, where

$$X(z) = \operatorname{Re}\left(\int_{z_0}^{z} \omega_1, \int_{z_0}^{z} \omega_2, \int_{z_0}^{x} \omega_3\right).$$

Here, z_0 is a fixed point of M. Moreover, if at any deleted point p_i, one of ω_1, ω_2, ω_3 has a pole of order at least 2, then S is also complete. The total curvature of S is

$$C(S) = -4\pi m,$$

where m is the degree of $\overline{g}$.

Proof. See, for example, [4, pp. 112–113] and [5, p. 82], Theorem 9.2. □

3. Weierstrass $\wp$ elliptic function

Let $L = [\omega_1, \omega_2]$ be a lattice $\mathbf{C}$. Associated to each L there is a doubly periodic meromorphic function, the Weierstrass $\wp$ function. It is

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left(\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2}\right),$$

where $\omega = m\omega_1 + n\omega_2$ for all $(m, n) \in \mathbf{Z} \times \mathbf{Z}$ and $(m, n) \neq (0, 0)$. It is easy to see $\wp$ is an even function. Moreover, we have

Lemma 1.

1. *$\wp^{(2k)}$ is an even function, $\wp^{(2k+1)}$ is an odd function, where $\wp^{(n)}$ denotes the* n*th derivative of $\wp$, $n \geq 0$.*
2. *$\wp^{(2k)} = P_k(\wp)$ and $\wp^{(2k+1)} = Q_k(\wp)\wp'$. Where P_k and Q_k are polynomials.*
3. *If $L = [1, \tau]$ and $\tau \in i\mathbf{R}$ then $\wp(\overline{z} + i) = \overline{\wp(z)}$ and $\wp(-\overline{z} + 1) = \overline{\wp(z)}$. Also,*

$$\int_0^1 [\wp^{(n)}]^2 \left(\frac{\tau}{2} + t\right) dt > 0,$$

for any $n \geq 0$. Furthermore,

$$\int_0^1 [\wp^{(n)}]^2 \left(\frac{\tau}{2} + t\right) \wp^m \left(\frac{\tau}{2} + t\right) dt$$

and

$$\int_0^1 \wp^m\left(\frac{\tau}{2}+t\right)dt$$

are nonzero real numbers, for any n, $m \in \mathbf{Z}$ *and* $n \geq 0$.

4. *If* $L = [1, i]$, $\omega = \frac{1+i}{2}$, *then* $\wp(\omega + i\bar{z}) = \overline{\wp(\omega+z)}$, $\wp(\omega - i\bar{z}) = -\overline{\wp(\omega+z)}$, *and* $\wp(w+\bar{z}) = \overline{\wp(\omega+z)}$. $\wp$ *has a double pole at 0, a double zero at* ω, *and no other zeros or poles. Furthermore,*

$$\wp^{(n)}\left(\frac{i}{2}+t\right) = -i^n\wp^{(n)}\left(\frac{1}{2}+it\right). \tag{2}$$

$$[\wp^{(n)}]^2\left(\frac{i}{2}+t\right) = (-1)^n[\wp^{(n)}]^2\left(\frac{1}{2}+it\right) \quad \textit{for } 0 \leq t \leq 1. \tag{3}$$

Proof. See [1, pp. 658 and 631], and [6, pp. 17–18]. □

Minimal surfaces of Enneper type with genus 1

Theorem 1. *There is a family of Enneper type surfaces of genus* 1 *and total curvature* $-8\pi(2n+1), n = 0, 1, 2, \cdots$.

Proof. We consider the square lattice $L = [1, i]$ in the complex plane $\mathbf{C}$. By Lemma 1, we have $\wp^{(2k+1)}(z) = Q_k(\wp(z))\wp'(z)$, where $Q_k(x)$ is a polynomial. Let $Q_k(x) = x^n R_k(x)$ such that $R_k(x) = a_0 + a_1 x + \cdots + a_l x^l$ and $a_0 \neq 0$. Then the degree of Q_k is $n + l = d$. We will denote $n = n(k)$ to emphasize that n depends on k. Since $\wp^{(2k+1)}$ has only one pole at 0 of order $2k+3$, and $Q_k(\wp)\wp'$ has only one pole at 0 of order $2d+3$; hence $d = k$ and $n(k) \leq k$. Let $M = T - \{0\} = \mathbf{C}/L - \{0\}$. We choose

$$g(z) = a\frac{\wp^{(2k+1)}(z)}{\wp^m(z)}, \tag{4}$$

where $a \in \mathbf{C}$ is an nonzero constant to be determined and $0 \leq n(k) < m \leq k+1$, $k \geq 0$. Then g has only poles at 0 of order $2(k-m)+3$ and at $\frac{1+i}{2}$ of order $2(m-n(k))-1$. Hence we know that the degree of g is $2(k-m)+3+2(m-n(k))-1 = 2(k-n(k))+2$. We choose

$$\eta = \wp^{2(m-n(k))-1}(z)\,dz. \tag{5}$$

Then η is never zero on M except at $\omega = \frac{1+i}{2}$, where g has a pole of order $l = 2(m-n(k))-1$, and η has an zero of order $4(m-n(k))-2 = 2l$. Let

$$\Psi = \frac{[\wp^{(2k+1)}]^2(z)}{\wp^{2n(k)+1}}\,dz = \frac{1}{a^2}g^2(z)\eta. \tag{6}$$

Since $\wp^{2(m-n(k))-1}(z)$ and $[\wp^{(2k+1)}]^2(z)/\wp^{2n(k)+1}(z)$ are both even functions, they have no residues at 0. Now in the Weierstrass representation formula with these g and η, we have

$$\begin{aligned} \omega_1 &= \frac{1}{2}(\eta - a^2\Psi), \\ \omega_2 &= \frac{i}{2}(\eta + a^2\Psi), \\ \omega_3 &= g\eta = a\wp^{(2k+1)}(z)\wp^{m-2n(k)-1}(z)\,dz \\ &= a\wp^{(2k+1)}(z)\wp^{m-2n(k)-1}(z)\,dz = a\wp^{m-n(k)-1}(z)R_k(\wp(z))\wp'(z)\,dz. \end{aligned} \tag{7}$$

Notice that ω_3 has a pole at 0 of order greater or equal to 3. By Proposition 1, this Weierstrass representation will generate an Enneper type minimal surface if equation (1) of Proposition 1 is satisfied. Since ω_3 is exact, we do not need to worry about its periods. Now let $\gamma_1(t) = i/2+t$, $\gamma_2(t) = 1/2+it$, $0 \le t \le 1$. Then γ_1 and γ_2 are generators of the fundamental group of $T = \mathbf{C}/L$. Hence it is enough to prove that we can choose an a such that

$$\operatorname{Re}\int_{\gamma i} \omega_j = 0,$$

for i, $j = 1, 2$. Notice that by Lemma 1 we have

$$[\wp^{(2k+1)}]^2\left(\frac{i}{2}+t\right) = -[\wp^{(2k+1)}]^2\left(\frac{1}{2}+it\right),$$

$$\wp^l\left(\frac{i}{2}+t\right) = (-1)^l\wp^l\left(\frac{1}{2}+it\right).$$

Hence

$$\begin{aligned} \int_{\gamma 2} \eta &= \int_0^1 \wp^{2(m-n(k))-1}\left(\frac{1}{2}+it\right) i\,dt \\ &= -i\int_0^1 \wp^{2(m-n(k))-1}\left(\frac{i}{2}+t\right) dt = -i\int_{\gamma 1} \eta, \end{aligned}$$

that is,

$$\int_{\gamma 2} \eta = -i\int_{\gamma 1} \eta. \tag{8}$$

Also

$$\begin{aligned} \int_{\gamma 2} \Psi &= \int_0^1 \frac{[\wp^{(2k+1)}]^2\left(\frac{1}{2}+it\right)}{\wp^{2n(k)+1}\left(\frac{1}{2}+it\right)} i\,dt \\ &= i\int_0^1 \frac{[\wp^{(2k+1)}]^2\left(\frac{i}{2}+t\right)}{\wp^{2n(k)+1}\left(\frac{i}{2}+t\right)}\,dt = i\int_{\gamma 1} \Psi, \end{aligned}$$

that is,

$$\int_{\gamma 2} \Psi = i\int_{\gamma 1} \Psi. \tag{9}$$

By Lemma 1, we know that $\int_{\gamma 1}\eta$ and $\int_{\gamma 1}\Psi$ are nonzero real numbers, so let $a^2 = \int_{\gamma 1}\eta / \int_{\gamma 1}\Psi$; then $a^2 \in \mathbf{R}$ and $a^2 \neq 0$. We have

$$\begin{aligned}
2\int_{\gamma 1}\omega_1 &= \int_{\gamma 1}\eta - a^2\int_{\gamma 1}\Psi = 0,\\
2\int_{\gamma 2}\omega_1 &= -i\int_{\gamma 1}\eta - ia^2\int_{\gamma 1}\Psi \in i\mathbf{R},\\
2\int_{\gamma 1}\omega_2 &= i\int_{\gamma 1}\eta + ia^2\int_{\gamma 1}\Psi \in i\mathbf{R},\\
2\int_{\gamma 2}\omega_2 &= i\int_{\gamma 2}\eta + ia^2\int_{\gamma 2}\Psi = \int_{\gamma 1}\eta - a^2\int_{\gamma 1}\Psi = 0.
\end{aligned}$$

Hence $\operatorname{Re}\int_{\gamma i}\omega_j = 0$. By Proposition 1 we get a complete minimal surface with genus 1 and one end. Since the degree of g is $2(k - n(k)) + 2$, the total curvature is $C(S) = -4\pi(2(k - n(k)) + 2)$. Let $d = k - n(k) + 1 \geq 1$. The only thing that remains to be proved is that d can be any odd positive integer. The next proposition will complete the proof of this theorem. □

Proposition 2. *The $n(k)$ defined in Theorem 2 satisfies*

$$n(k) = \begin{cases} 0 & \text{if } k \text{ is even},\\ 1 & \text{if } k \text{ is odd}.\end{cases} \tag{10}$$

Proof. First we look at the formula in Lemma 1 stating that

$$\wp^{(2k)} = P_k(\wp), \qquad \wp^{(2k+1)} = Q_k(\wp)\wp'.$$

P_k and Q_k are polynomials. Let $Q_k(x) = a_0^k + a_1^k x + \cdots + a_k^k x^k$. We claim that

1. $a_j^k = 0$ if $k \not\equiv j \bmod 2$, and $a_j^k \in \mathbf{R}$,
2. $a_0^{2k} \neq 0$, $a_1^{2k+1} \neq 0$.

Clearly claim 1 and claim 2 imply the proposition.

Now $\wp'^2 = 4\wp^3 - g_2\wp - g_3$, where g_2 and g_3 depend on the lattice $L = [1, \tau]$. When $\tau = i$, we have $g_3 = 0$, and $g_2 \in \mathbf{R}$, $g_2 \neq 0$ (see [3], Corollary 3, p. 40). We prove claim 1 by induction. Since $\wp' = 1 \cdot \wp'$, so $Q_0 = 1$, and $\wp'' = 6\wp^2 - g_2/2$, $\wp''' = 12\wp\wp'$, so that $Q_1(x) = 12x$. Thus claim 1 is true for $k = 0$ and $k = 1$. Suppose claim 1 is true for $k = 2n$ and $2n + 1$, $n \geq 0$. Then

$$\wp^{(4n+1)} = \wp' \sum_{j=0}^{n} a_{2j}^{2n}\wp^{2j}, \wp^{(4n+3)} = \wp' \sum_{j=0}^{n} a_{2j+1}^{2n+1}\wp^{2j+1}.$$

We have that

$$\begin{aligned}\wp^{(4n+4)} &= \wp'^2 \sum_{j=0}^{n} (2j+1) a_{2j+1}^{2n+1} \wp^{2j} + \wp'' \sum_{j=0}^{n} a_{2j+1}^{2n+1} \wp^{2j+1} \\ &= (4\wp^3 - g_2\wp) \sum_{j=0}^{n} (2j+1) a_{2j+1}^{2n+1} \wp^{2j} + (6\wp^2 - g_2/2) \sum_{j=0}^{n} a_{2j+1}^{2n+1} \wp^{2j+1} \\ &= \sum_{j=0}^{n} (4(2j+1)+6) a_{2j+1}^{2n+1} \wp^{2j+3} - \frac{g_2}{2} \sum_{j=0}^{n} (2(2j+1)+1) a_{2j+1}^{2n+1} \wp^{2j+1}.\end{aligned}$$

Hence,

$$\begin{aligned}\wp^{(4n+5)} &= \wp' \left\{ \sum_{j=0}^{n} (2j+3)(8j+10) a_{2j+1}^{2n+1} \wp^{2j+2} \right. \\ &\qquad \left. - \frac{g_2}{2} \sum_{j=0}^{n} (2j+1)(4j+3) a_{2j+1}^{2n+1} \wp^{2j} \right\} \\ &= \wp' \left\{ -\frac{3g_2}{2} a_1^{2n+1} + \sum_{j=1}^{n} (2j+1) \left[(8j+2) a_{2j-1}^{2n+1} - \frac{g_2}{2}(4j+3) a_{2j+1}^{2n+1} \right] \wp^{2j} \right. \\ &\qquad \left. + (2n+3)(8n+10) a_{2n+1}^{2n+1} \wp^{2n+2} \right\} \\ &= \wp' \sum_{j=0}^{n+1} a_{2j}^{2n+2} \wp^{2j}.\end{aligned}$$

Since there are only even terms and all the a_j^{2n+1} and j are real, so a_j^{2n+2} is real and claim 1 is true for $k = 2n+2$. Also we can see by the computation that $a_0^{2n+2} = -\frac{3}{2} g^2 a_1^{2n+1}$. Similarly,

$$\begin{aligned}\wp^{(4n+6)} &= \wp'^2 \sum_{j=0}^{n+1} 2j a_{2j}^{2n+2} \wp^{2j-1} + \wp'' \sum_{j=0}^{n+1} a_{2j}^{2n+2} \wp^{2j} \\ &= \sum_{j=0}^{n+1} (8j+6) a_{2j}^{2n+2} \wp^{2j+2} - \frac{g_2}{2} \sum_{j=0}^{n+1} (8j+1) a_{2j}^{2n+2} \wp^{2j} = \sum_{j=0}^{n+1} b_{2j+2} \wp^{2j+2},\end{aligned}$$

so

$$\wp^{(4n+7)} = \wp' \sum_{j=0}^{n+1} (2j+2) b_{2j+2} \wp^{2j+1} = \wp' \sum_{j=0}^{n+1} a_{2j+1}^{2n+3} \wp^{2j+1}.$$

Hence $k = 2n+3$ is true for claim 1. This completes the proof of claim 1.

Notice that $a_0^0 = 1$, $a_1^1 = 12$, and $a_0^{2n} = -\frac{3g_2}{2} a_1^{2n-1}$ for $n \geq 1$. So we need only to prove that $a_0^{2n} \neq 0$. Let $\omega = \frac{1+i}{2}$. Since $\wp(\omega) = \wp'(\omega) = 0$,

$\wp''(\omega) \neq 0$, so ω is a double zero of $\wp$, a single zero of $\wp'$. In (2) of Lemma 1 set $t = 1/2$ and $n = 4k$. We get $\wp^{(4k)}(\omega) = -\wp^{(4k)}(\omega)$, so $\wp^{(4k)}(\omega) = 0$. Also by $\wp^{(2k+1)} = \wp' Q_k(\wp)$ we know that $\wp^{(2k+1)}(\omega) = \wp^{(2k+1)}(\omega + \frac{1}{2}) = 0$. If $a_0^{2n} = 0$, then ω will be a zero of $\wp^{(4n+1)}$ of order at least 3. So if we prove that $\wp^{(4n+2)}(\omega) \neq 0$, then $\wp^{(4n+1)}$ will have only a single zero at ω and thus $a_0^{2n} \neq 0$. We will count the number of zeros for $\wp^{(4n+2)}$. Let I be the open interval $(0, 1/2)$. Then on $\omega + I$ (for $\omega + I$ we mean the interval $1/2 + i/2 + t$, $0 < t < 1/2$, similarly $\omega + iI$ means the interval $1/2 + i/2 + it$, $0 < t < 1/2$), $\wp$ and $\wp'$ are real, and $\wp(\omega + \overline{z}) = \overline{\wp(\omega + z)}$, $\wp'(\omega + \overline{z}) = \overline{\wp'(\omega + z)}$. By claim 1, any $\wp^{(k)}$, $k \geq 0$, has these properties. If $z = x + iy$, then

$$\left. \frac{\partial \operatorname{Re} \wp^{(k)}(\omega + z)}{\partial y} \right|_{y=0} = 0$$

on $\omega + I$. For a holomorphic function f,

$$f' = 2(\operatorname{Re} f)_z = \frac{\partial \operatorname{Re} f}{\partial x} - i \frac{\partial \operatorname{Re} f}{\partial y},$$

on $\omega + I$ we have

$$\wp^{(k+1)}(\omega + x) = \frac{\partial \operatorname{Re} \wp^{(k)}(\omega + x)}{\partial x} - i \frac{\partial \operatorname{Re} \wp(\omega + x)}{\partial y} = \frac{\partial \wp^{(k)}(\omega + x)}{\partial x}$$

for all $x \in I$. We claim that for any $n \geq 0$, $\wp^{(4n+2)}$ has at least $n+1$ different zeros on $\omega + I$. Since $\wp'(\omega + 1/2) = \wp'(\omega) = 0$, by Rolle's theorem there is at least one $x_0 \in I$ such that $\wp''(\omega + x_0) = 0$. Hence for $n = 0$ the claim is true. We now apply induction. Suppose for $n = k \geq 0$, $\wp^{(4k+2)}$ has at least $k + 1$ different zero points on $\omega + I$. Then by Rolle's theorem, $\wp^{(4k+3)}$ has at least k different zero points on $\omega + I$. Note $\wp^{(4k+3)}(\omega) = \wp^{(4k+3)}(\omega + 1/2) = 0$, so on the closure of $\omega + I$, $\wp^{(4k+3)}$ has at least $k + 2$ different zeros. Again by Rolle's theorem $\wp^{(4k+4)}$ has at least $k + 1$ different zeros on $\omega + I$. Because $\wp^{(4k+4)}(\omega) = 0$, on the closure of $\omega + I$, $\wp^{(4k+4)}$ has at least $k + 2$ different zeros. Hence $\wp^{(4k+5)}$ has at least $k + 1$ different zeros on $\omega + I$. Again, $\wp^{(4k+5)}(\omega) = \wp^{(4k+5)}(\omega + 1/2) = 0$, so that $\wp^{(4k+6)}$ has at least $k + 2$ different zeros on $\omega + I$. Hence we have thus proved this claim. For $x \in I$, $\wp(\omega + x) = \wp(\omega - x)$, $\wp(\omega + x) = \wp(\omega + ix)$, $\wp(\omega + ix) = \wp(\omega - ix)$. Hence $\wp^{(4n+2)}$ has at least $4n + 4$ different zeros. But $\deg \wp^{(4n+2)} = 4n + 4$, so $\wp^{(4n+2)}$ can have only $4n + 4$ zeros. Thus we have found all of the zeros of $\wp^{(4n+2)}$. Because these zeros are in $\omega + I$, $\omega - I$, $\omega + iI$, $\omega - iI$, we conclude that $\wp^{(4n+2)}(\omega) \neq 0$. Thus the proof of this proposition is complete. □

Remark 1. When setting $k = 0$, we get Chen and Gackstäter's genus 1 example.

Remark 2. By the proof of Proposition 2, we get some properties of the Weierstrass $\wp$ function associated to $L = [1, i]$. We list these properties as a separate theorem.

Theorem 2. *The Weierstrass $\wp$ function associated with* $[1, i]$ *has the following properties*:

1. *For* $n \geq 0$, *all the zeros of* $\wp^{(n)}$ *are in the two lines* $\gamma_1(t) = i/2 + t$, $\gamma_2(t) = 1/2 + it$, $0 \leq t \leq 1$. *The zeros are symmetric about* $\omega = \frac{1+i}{2}$.
2. *For* $n \geq 0$, $\wp^{(4n)}$ *has a double zero at* ω, $\wp^{(4n+3)}$ *has a triple zero at* ω. *Any other zeros of* $\wp^{(n)}$ *are simple.*
3. $\wp^{(4n)}(\omega) = \wp^{(4n+1)}(\omega) = \wp^{(4n+3)}(\omega) = 0$ *and* $\wp^{(4n+2)}(\omega) \neq 0$ *for* $n \geq 0$.

Proof. Just as in the proof of Proposition 1 count the number of zeros on $\omega + I$, using the three symmetries and $\deg \wp^{(k)} = k + 2$. Note that because $\wp^{(4n-1)}(\omega) = \wp^{(4n)}(\omega) = \wp^{(4n+1)}(\omega) = 0$, $\wp^{(4n+2)}(\omega) \neq 0$, we get claim 2. □

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