PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 108, Number 4, April 1990

A TOTALLY REAL THREE-SPHERE IN $\mathbf{C}^3$ BOUNDING A FAMILY OF ANALYTIC DISKS

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(Communicated by Irwin Kra)

ABSTRACT. We construct a smoothly embedded totally real three-sphere S in $\mathbf{C}^3$ and a one-parameter family of analytic disks in $\mathbf{C}^3$ that have boundaries in S.

1. INTRODUCTION

Denote by D the open unit disk $\{z \in \mathbf{C}: |z| < 1\}$ in $\mathbf{C}$, by $\overline{D}$ the closed unit disk and by bD its boundary $\{z \in \mathbf{C}: |z| = 1\}$. Let $A(D)$ be the algebra of all continuous functions on $\overline{D}$ that are holomorphic on D. An *analtyic disk* with boundary in a subset $M \subset \mathbf{C}^n$ is a map $f = (f_1, \dots, f_n): \overline{D} \to \mathbf{C}^n$, $f_j \in A(D)$ $(j = 1, \dots, n)$, such that $f(bD)$ is contained in M.

Recall that a real submanifold M of $\mathbf{C}^n$ of class $\mathbf{C}^1$ is called *totally real* if for each $x \in M$ the tangent space T_xM of M at x contains no nontrivial complex subspace, i.e., $T_xM \cap iT_xM = \{0\}$. In this note we shall construct an embedded totally real three-sphere S in $\mathbf{C}^3$ which bounds a one-parameter family of analytic disks. More precisely, we prove

Theorem 1. *There is a smooth totally real submanifold S of $\mathbf{C}^3$ diffeomorphic to $\{x \in \mathbf{R}^4: |x| = 1\}$ and an embedding $F: \overline{D} \times [-1, 1] \to \mathbf{C}^3$ such that for each $t \in [-1, 1]$ the map $z \to F(z, t)$, $z \in \overline{D}$, is an analytic disk with boundary in S.*

The first explicit totally real embedding of the real three-sphere into $\mathbf{C}^3$ was given by Ahern and Rudin [2]. The existence of such embeddings also follows from a theorem of Gromov [7, 8, p. 193, 5, Theorem 1.4, 6]. However, it seems difficult to find analytic disks with boundaries in a given submanifold; we do not know whether there are any such disks in the example of Ahern and Rudin [2].

We believe that our example is of interest for the following reason. If $M \subset \mathbf{C}^n$ is a smoothly embedded compact *lagrange* submanifold of $\mathbf{C}^n$, i.e., the

Received by the editors October 3, 1986 and, in revised form, April 10, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 32E20; Secondary 57R40.
Research supported in part by a grant from Rāziskovalna skupnost SR Slovenije.

pullback of the 2-form $\omega = \sum_{j=1}^{n} dz_j \wedge d\overline{z}_j$ to M vanishes, then for every nonconstant analytic disk f with boundary in M the curve $f\colon bD \to M$ represents a nontrivial class in the homology group $H_1 M$. To see this assume on the contrary that this path bounds a 2-cycle σ in M. By a theorem of Čirka [4, p. 293] f is smooth on $\overline{D}$, and the Stokes's theorem applied to the one-form $\alpha = \sum_{j=1}^{n} z_j \, d\overline{z}_j$ yields

$$0 = \int_\sigma \omega = \int_\sigma d\alpha = \int_{f(bD)} \alpha = \int_{bD} f^* \alpha = \int_D f^* \omega.$$

However, since $f^*\omega = |f'|^2 \, dz \wedge d\overline{z}$ on D, the last integral is nonzero, a contradiction. This argument was communicated to me by L. Lempert who raised the question whether the same is true if M is only totally real. (Recall that every lagrange submanifold $M \subset \mathbf{C}^n$ is also totally real.) Our theorem shows that this is not the case: the three-sphere S is simply connected, so $H_1 S = 0$, and yet it may bound analytic disks in $\mathbf{C}^3$.

It is not known whether the three-sphere S admits a lagrange embedding into $\mathbf{C}^3$. In fact it was conjectured that no compact simply connected n-dimensional manifold M admits a lagrange embedding into $\mathbf{C}^n$. As for the immersions, every totally real immersion of a compact n-manifold M into $\mathbf{C}^n$ is regularly homotopic through totally real immersions to a lagrange immersion of M into $\mathbf{C}^n$ [10, 7, 8, p. 61].

Denote by $\hat{M}$ the polynomially convex hull of a set $M \subset \mathbf{C}^n$. If M is a compact embedded totally real submanifold of $\mathbf{C}^n$ of real dimension n, it is known [1] that $\hat{M}$ has topological dimension at least $n+1$. It would be of interest to know whether every such M bounds analytic disks or analytic varieties in $\mathbf{C}^n$. If so, is $\hat{M} \backslash M$ the union of closed analytic subvarieties of $\mathbf{C}^n \backslash M$? Every such subvariety with no zero-dimensional components is contained in the hull of M according to the maximum principle. Note that the general technique of constructing analytic disks due to Bishop [3] and Hill and Taiani [9] does not apply in the totally real case that we are dealing with. Important results in this direction were obtained by Gromov [11].

We shall prove Theorem 1 in §2. In the construction of S we will need an extension theorem for functions that do not annihilate a zero-free complex vector field at any point of an n-dimensional cube (Theorem 3). This result of independent interest can be proved by different methods; in §3 we shall prove it using techniques of Gromov. (See [7], §2.4 in [8] and also the exposition in [5].)

I wish to thank L. Lempert for proposing this problem. Thanks also go to S. Webster and E. L. Stout.

2. Proof of Theorem 1

We begin with

Lemma 2. *There exists a real-analytic function* $g : \mathbf{C} \to \mathbf{C}$ *such that the submanifold* $N = \{(z, g(z)) : z \in \mathbf{C}\}$ *of* $\mathbf{C}^2$ *is totally real and* $g(z) = 0$ *for each* $z \in bD$.

Proof. The manifold N is totally real when the derivative $\partial g/\partial \overline{z}$ has no zeroes on $\mathbf{C}$. Instead of simply giving the formula (2.1) for g we will show how to find such a function.

We set $g(z) = (z\overline{z} - 1)h(z)$ in order to have $g(z) = 0$ when $|z| = 1$. Then

$$\partial g/\partial \overline{z}(z) = zh(z) + (z\overline{z} - 1)\partial h/\partial \overline{z}(z).$$

When $|z| = 1$, $\partial g/\partial \overline{z}(z) = zh(z)$. Since $\partial g/\partial \overline{z}$ is zero free on $\mathbf{C}$, its winding number on the circle $bD = \{|z| = 1\}$ equals zero, so the winding number of h on bD is -1. To achieve this we set $h(z) = \overline{z}e^{k(z)}$. Then

$$\partial g/\partial \overline{z}(z) = e^{k(z)}((2z\overline{z} - 1) + (z\overline{z} - 1)\overline{z}\partial k/\partial \overline{z}(z)).$$

If we choose $k(z) = iz\overline{z}$, $\partial k/\partial \overline{z}(z) = iz$, and set $t = z\overline{z}$, the expression in the parentheses equals $(2t - 1) + i(t - 1)t$ which does not vanish for any real t. Thus the function

$$g(z) = (z\overline{z} - 1)\overline{z}e^{iz\overline{z}} \tag{2.1}$$

satisfies Lemma 2. This concludes the proof.

Choose any smooth function $h : R \to [0, \infty)$ which equals 0 on $(-\infty, 2]$ and is strictly convex on $(2, \infty)$. The real hypersurface $\Gamma \subset \mathbf{C}^2$ defined by

$$\Gamma = \{(z, w) \in \mathbf{C}^2 : r(z, w) = h(z\overline{z} + u^2) + (v - 1)^2 = 1\} \tag{2.2}$$

(here $w = u + iv$) is smooth, diffeomorphic to the real three-sphere, and it contains the set $\overline{D} \times [-1, 1]$.

Let g be as in Lemma 2. Define $f(z, w) = g(z) + w$ on $z \in \overline{D}$, $w \in [-1, 1]$ and extend f smoothly to $\mathbf{C}^2$. We will show that the extension of f to Γ can be chosen in such a way that the real submanifold

$$S = \{(z, w, f(z, w)) \in \mathbf{C}^3 : (z, w) \in \Gamma\} \tag{2.3}$$

of $\mathbf{C}^3$ satisfies the conclusion of Theorem 1.

Clearly S is a smoothly embedded three-sphere which bounds the one parameter family of analytic disks $F : \overline{D} \times [-1, 1] \to \mathbf{C}^3$, $F(z, t) = (z, t, t)$ since g vanishes on $|z| = 1$.

It remains to show that S is totally real for an appropriate choice of f. Let L be the tangential $\overline{\partial}$ operator on Γ. Then S is totally real if and only if Lf has no zeros on Γ. We have

$$\begin{aligned} Lf(z, w) &= \partial r/\partial \overline{z}\partial f/\partial \overline{w} - \partial r/\partial \overline{w}\partial f/\partial \overline{z} \\ &= h'(z\overline{z} + u^2)z\partial f/\partial \overline{w} - (h'(z\overline{z} + u^2)u + i(v - 1))\partial f/\partial \overline{z}. \end{aligned}$$

On the set $(z, w) \in \overline{D} \times [-1, 1]$ we have $h'(z\overline{z} + u^2) = 0$, so $Lf = i\partial g/\partial \overline{z}$ which is nonvanishing. Thus the part of S lying over $\mathbf{D} \times [-1, 1]$ is totally real. There is an open subset U of Γ containing $\overline{D} \times [-1, 1]$ such that Lf has no zeroes on $\overline{U}$ and $\Gamma \backslash U$ is diffeomorphic to a closed three dimensional cube $I^3 \subset \mathbf{R}^3$. According to Theorem 3 in §3 there is an extension of f from $\overline{U}$ to Γ such that Lf has no zeroes on Γ. For such f the manifold S given by (2.3) is totally real and Theorem 1 is proved.

3. Functions not annihilating a complex vector field

We denote by I^n the closed n-dimensional cube $[0, 1]^n$ in $\mathbf{R}^n$. Let $x = (x_1, \dots, x_n)$ be real coordinates on $\mathbf{R}^n$. A complex vector field on I^n with continuous coefficients is an expression $L = \sum_{j=1}^n a_j(x)\,\partial/\partial x_j$, where a_j are continuous complex functions on I^n. If f is a complex $\mathbf{C}^1$ function on I^n, then $Lf(x) = \sum_{j=1}^n a_j(x)\,\partial f/\partial x_j(x)$. The vector field L is zero-free on I^n if for each $x \in I^n$ at least one number $a_j(x)$ is nonzero.

Theorem 3. *Let L be a zero-free complex vector field with continuous coefficients on I^n $(n \neq 2)$. For each $\mathbf{C}^1$ function f_0 on I^n such that Lf_0 is zero-free on the boundary of I^n there is a $\mathbf{C}^1$ function f on I^n which coincides with f_0 near the boundary of I^n such that Lf is zero-free on all of I^n.*

Remark. Theorem 3 is false for $n = 2$ as the following example shows. Take $L = \partial/\partial\overline{z} = (\partial/\partial x + i\,\partial/\partial y)/2$ and $f(z) = z\overline{z} = x^2 + y^2$. Then $Lf(z) = z$ is nonvanishing on the boundary of $[-1, 1]^2$, but it can not be extended to a nonvanishing function on $[-1, 1]^2$ since it has positive winding number.

Proof. This result follows from Gromov's Lemma 3.1.3 in [7]. See also §2.4 in [8]. Let $L = L_1 + iL_2$, where $L_1 = \sum_{j=1}^n a_j\,\partial/\partial x_j$ and $L_2 = \sum_{j=1}^n b_j\,\partial/\partial x_j$ are real-valued vector fields on I^n. If $f = u + iv$, then $Lf = (L_1 u - L_2 v) + i(L_1 v + L_2 u)$. We associate to a function $f = u + iv$ the section $x \to (x; u(x), v(x))$ $(x \in I^n)$ of the product bundle $\pi : X = I^n \times \mathbf{R}^2 \to I^n$. Let X^1 be the manifold of one-jets of sections of the bundle $X \to I^n$. X^1 is isomorphic to the product $X \times R^{2n}$; the point $(\alpha, \beta) \in \mathbf{R}^{2n}$ corresponding to a section $x \to (x; u(x), v(x))$ of X is determined by $\alpha_j = \partial u/\partial x_j$, $\beta_j = \partial v/\partial x_j$ $(1 \le j \le n)$.

Let $\Omega \subset X^1$ be the set of all points $(x; q; \alpha, \beta)$ in X^1 $(x \in I^n,\ q \in \mathbf{R}^2,\ \alpha, \beta \in \mathbf{R}^n)$ for which at least one of the real numbers

$$\sum_{i=1}^n a_i(x)\alpha_i - b_i(x)\beta_i,$$

$$\sum_{i=1}^n b_i(x)\alpha_i + a_i(x)\beta_i \tag{3.1}$$

is nonzero. In Gromov's terminology the set Ω is an *open differential relation of order one* on the bundle $\pi: X \to I^n$.

Lemma 4. *The relation Ω defined above is ample in the coordinate directions $x_1, \ldots, x_n$ on I^n.*

Note. For the definition of ampleness see [7, p. 331] or §2 in [5] or [8, p. 180].

Proof. By symmetry it suffices to prove ampleness in the coordinate direction x_1. Fix a point $x^0 \in I^n$, $q^0 \in \mathbf{R}^2$, $\alpha' = (\alpha_2, \ldots, \alpha_n)$, $\beta' = (\beta_2, \ldots, \beta_n)$ and consider the set

$$\Omega' = \{(\alpha_1, \beta_1) \in \mathbf{R}^2 : (x^0; q^0; \alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots) \in \Omega\}.$$

To prove that Ω is ample in the direction x_1 we must show that either Ω' is empty or else the convex hull of each of its connected components in $\mathbf{R}^2$ equals all of $\mathbf{R}^2$.

If $a_1(x^0) = b_1(x^0) = 0$, then (3.1) shows that Ω' is either empty or $\mathbf{R}^2$. If on the other hand at least one of the numbers $a_1(x^0), b_1(x^0)$ is nonzero, then the system of linear equations

$$a_1(x^0)\alpha_1 - b_1(x^0)\beta_1 = c,$$
$$b_1(x^0)\alpha_1 + a_1(x^0)\beta_1 = d$$

has determinant $a_1(x^0)^2 + b_1(x^0)^2 > 0$ whence it has precisely one solution for each $(c, d) \in \mathbf{R}^2$. In this case Ω' is the complement of a point in $\mathbf{R}^2$. This proves that Ω is ample.

Lemma 5. *If $n \neq 2$ and if α_j, β_j are continuous real-valued functions on the boundary of I^n such that the expression*

$$(3.2)\quad F(x) = \sum_{j=1}^{n} \left(a_j(x)\alpha_j(x) - b_j(x)\beta_j(x)\right) + i \cdot \sum_{j=1}^{n} \left(b_j(x)\alpha_j(x) + a_j(x)\beta_j(x)\right)$$

is zero-free on the boundary of I^n, then there exist continuous extensions of α_j, β_j $(1 \le j \le n)$ to I^n such that F is zero-free on I^n.

Proof. We claim that for $n \neq 2$ the map $F : bI^n \to \mathbf{C}\backslash\{0\}$ can be extended to a map $F : I^n \to \mathbf{C}\backslash\{0\}$. If $n = 1$ this holds because $\mathbf{C}\backslash\{0\}$ is path connected. If $n > 1$, the obstruction to extending F is an element of the group $\pi_{n-1}(\mathbf{C}\backslash\{0\}) = \pi_{n-1}(S^1)$ which is trivial when $n - 1 \ge 2$. We fix such an extension of F to I^n.

We subdivide the cube I^n into smaller closed cubes $I_1, \ldots, I_r$ with faces parallel to the coordinate axes such that two distinct cubes have at most a face in common and for each I_k there is an index j_k for which

$$(3.3)\qquad a_{j_k}(x)^2 + b_{j_k}(x)^2 > 0, \qquad x \in I_k.$$

We now perform stepwise extension of the functions α_j and β_j to the cubes I_k. On I_k we extend α_j, β_j, $j \neq j_k$, arbitrarily, without changing their values on those faces of I_k where they have been defined in previous steps. Because of (3.3) the values of α_{j_k} and β_{j_k} on I_k are now uniquely determined by (3.2). In a finite number of steps we find the desired extensions and Lemma 5 is proved.

We can now conclude the proof of Theorem 3. Let $f_0 = u_0 + iv_0$ be as in the statement of the theorem. Set $\alpha_j = \partial u_0(x)/\partial x_j$ and $\beta_j = \partial v_0(x)/\partial x_j$ for $x \in bI^n$ and $1 \le j \le n$. We extend the functions α_j, β_j $(1 \le j \le n)$ to I^n using Lemma 5. (Note that $F(x) = Lf_0(x) \neq 0$ for $x \in bI^n$.) The map $\varphi : I^n \to X^1$, $\varphi(x) = (x; f_0(x); \alpha(x), \beta(x))$ is a section of the relation Ω over I^n which coincides with the one-jet $j^1 f_0$ of the section $x \to (x; f_0(x))$ on the boundary bI^n. Since Ω is ample by Lemma 4, Gromov's Lemma 3.1.3 in [7] implies that there is a $\mathbf{C}^1$ function $f : I^n \to \mathbf{C}$ whose one-jet $j^1 f$ is a section of Ω over I^n and $j^1 f = j^1 f_0$ on bI^n. This means precisely that $Lf(x)$ is nonvanishing on I^n and $f = f_0$ on bI^n. Theorem 3 is proved.

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