A LINEARIZATION OF THE CIRCULAR MAXIMAL OPERATOR

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Abstract. An interesting linearization of the circular maximal operator is of restricted weak type (2,2).

The spherical maximal operator $\mathcal{M}$ on $\mathbb{R}^n$ is defined by

$$\mathcal{M}f(x) = \sup_{r>0} \int_{|y|=1} |f|(x - ty) \, dy.$$ 

Here $f$ is an appropriate function and $dy$ denotes normalized Lebesgue measure on the unit sphere in $\mathbb{R}^n$. In [4] Stein proved that $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^n)$ if $p > n/(n-1)$ and $n \geq 3$. More recently Bourgain [1] established the same result for $n = 2$. Bourgain [2] also noted that when $n \geq 3$ and $p = n/(n-1)$, $\mathcal{M}$ is restricted weak type $(p, p)$—that is, $\mathcal{M}$ maps $L^{p,1}(\mathbb{R}^n)$ into $L^{p,\infty}(\mathbb{R}^n)$. This result implies that of [4]. It is then natural to ask if $\mathcal{M}$ maps $L^{2,1}(\mathbb{R}^2)$ into $L^{2,\infty}(\mathbb{R}^2)$. Leckband [3] provided a partial result: the answer is yes if one restricts to the subspace of radial functions. The purpose of this note is to give a partial result with a different flavor. We restrict the operator instead of its domain and define, as in [1, p. 70], a linearization $T$ of $\mathcal{M}$ by

$$Tf(x) = \int_{|y|=1} f(x - |x|y) \, dy.$$ 

Theorem. The operator $T$ maps $L^{2,1}(\mathbb{R}^2)$ into $L^{2,\infty}(\mathbb{R}^2)$.

In all known cases the mapping properties of $T$ are as bad as those of $\mathcal{M}$, and so this theorem lends support to the conjecture that $\mathcal{M}$ is of restricted weak type $(2, 2)$ on $\mathbb{R}^2$.

For $x \in \mathbb{R}^2$, let $L_x$ be the line through $x$ perpendicular to the radial segment from the origin to $x$. Define an operator $S$ by

$$Sf(x) = \frac{1}{|x|} \int_{L_x} f(y) \, dy,$$

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where $dy$ is one-dimensional Lebesgue measure on $L_x$. Then $S$ is a weighted version of the Radon transform on $\mathbb{R}^2$ and (see Lemma 2) is equivalent to the adjoint of $T$. Our theorem follows from the boundedness of $S$ from $L^2,1(\mathbb{R}^2)$ to $L^{2,\infty}(\mathbb{R}^2)$, which is proved as Lemma 4. To simplify notation we regard points in $\mathbb{R}^2$ as complex numbers.

**Lemma 1.** If $z \neq 0$ is a complex number, define $z' = ze^{-i\pi/2}/|z|$. If $g$ and $h$ are nonnegative and measurable, then

$$
\int_0^{2\pi} \int_0^{2\pi} h(e^{i\theta} + e^{i\xi})g(e^{i\xi'})d\theta d\xi
= \int_{|z|<2} h(z) \left[ g \left( \frac{z}{2} + z' \sqrt{1 - \frac{|z|^2}{4}} \right) + g \left( \frac{z}{2} - z' \sqrt{1 - \frac{|z|^2}{4}} \right) \right] \frac{dz}{|z|\sqrt{1 - \frac{|z|^2}{4}}},
$$

where $dz$ denotes two-dimensional Lebesgue measure.

**Proof.** If $z = e^{i\theta} + e^{i\xi}$ with $\theta < \xi < \theta + \pi$, then a sketch shows that

$$
e^{i\xi} = \frac{z}{2} + z' \sqrt{1 - \frac{|z|^2}{4}}.
$$

Also the Jacobian of the map

$$(t, \theta) \mapsto z = e^{i\theta} + e^{i\xi}$$

is

$$|\sin(\xi - \theta)| = |z|\sqrt{1 - \frac{|z|^2}{4}}.$$

Thus the formula

$$
\int_0^{2\pi} \int_0^{\theta + \pi} h(e^{i\theta} + e^{i\xi})g(e^{i\xi'})d\theta d\xi
= \int_{|z|<2} h(z) g \left( \frac{z}{2} + z' \sqrt{1 - \frac{|z|^2}{4}} \right) \frac{dz}{|z|\sqrt{1 - \frac{|z|^2}{4}}},
$$

is just a change of variable. A similar formula for the range $\theta + \pi < \xi < \theta + 2\pi$ completes the proof. \(\Box\)

**Lemma 2.** If $g$ and $h$ are nonnegative and measurable, then

$$
\int_{\mathbb{R}^2} Tf(x)g(x)dx = \frac{1}{2} \int_{\mathbb{R}^2} f(x)Sg \left( \frac{x}{2} \right)dx.
$$
Proof.

\[
\int_{\mathbb{R}^2} T_f(x)g(x) \, dx = \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} f(re^{i\theta} + re^{i\theta})g(re^{i\theta})r \, dt \, d\theta \, dr
\]

\[
= \int_0^\infty \int_0^{2\pi} f(re^{i\phi}) \left\{ g \left( r \left[ \frac{ue^{i\phi}}{2} + e^{i(\phi - \frac{\pi}{4})} \sqrt{1 - \frac{u^2}{4}} \right] \right) 
+ g \left( r \left[ \frac{ue^{i\phi}}{2} - e^{i(\phi - \frac{\pi}{4})} \sqrt{1 - \frac{u^2}{4}} \right] \right) \right\} r \, d\phi \, du \, dr
\]

by Lemma 1, with \( z = eu^{i\phi} \). To this last expression apply Fubini's theorem, let \( s = ur \), and apply Fubini's theorem again. The result is

\[
\int_0^\infty \int_0^{2\pi} f(se^{i\phi}) \left\{ g \left( \frac{s}{2} e^{i\phi} + \frac{s}{2} e^{i(\phi - \frac{\pi}{4})} \sqrt{1 - \frac{u^2}{4}} \right) 
+ g \left( \frac{s}{2} e^{i\phi} - \frac{s}{2} e^{i(\phi - \frac{\pi}{4})} \sqrt{1 - \frac{u^2}{4}} \right) \right\} s \, d\phi \, du \, ds
\]

\[
= \int_0^\infty \int_0^\infty \int_0^{2\pi} f(se^{i\phi}) \left\{ g \left( \frac{s}{2} e^{i\phi} + sve^{i(\phi - \frac{\pi}{4})} \right) 
+ g \left( \frac{s}{2} e^{i\phi} - sve^{i(\phi - \frac{\pi}{4})} \right) \right\} s \, d\phi \, dv \, ds
\]

where the equalities are from the changes of variable

\[
v = \sqrt{1 - \frac{u^2}{4}} \quad \text{and} \quad t = sv.
\]

This last integral is

\[
\frac{1}{2} \int_{\mathbb{R}^2} f(x) S g \left( \frac{x}{2} \right) \, dx.
\]

In what follows, \(| \cdot |\) will denote Lebesgue measure on either \( \mathbb{R} \) or \( \mathbb{R}^2 \), the exact meaning being clear from the context. Also, \( \chi(x, E) \) will stand for the characteristic function of the set \( E \) evaluated at \( x \).

**Lemma 3.** Suppose \( h \) and \( k \) are measurable functions on \([0, \infty]\) with \( 0 < h, k \leq 1 \). Then

\[
\int_0^\infty \int_0^\infty \min(h(s), k(r)) \, ds \, dr \leq 4 \left( \int_0^\infty sh(s) \, ds \right)^{\frac{1}{2}} \left( \int_0^\infty rk(r) \, dr \right)^{\frac{1}{2}}.
\]

**Proof.** The left hand side of the conclusion is not affected by measure-preserving rearrangements of \( h \) and \( k \) while the right hand side will be least when \( h \) and \( k \)
are decreasing. So replacing $h$ and $k$ by suitable approximations shows that it is enough to establish the lemma under the additional hypotheses that $h$ and $k$ are continuous positive strictly decreasing functions satisfying $h(0) = k(0) = 1$. Now
\[
\int_0^\infty \int_0^\infty \min(h(s), k(r)) \, ds \, dr = \int_0^\infty h(s)\{r : k(r) > h(s)\} \, ds \\
+ \int_0^\infty k(r)\{s : h(s) > k(r)\} \, dr = I_1 + I_2.
\]
Since $\{|r : k(r) > h(s)\}| = k^{-1}(h(s))$, $I_1 = \int_0^\infty h(s)k^{-1}(h(s)) \, ds = \int_0^\infty \int_0^{k^{-1}(h(s))} \chi(y, [0, h(s)]) \, dy \, dx \, ds$
\[
= \int_0^\infty \int_0^{k^{-1}(x)} \int_0^\infty \chi(x, [0, k^{-1}(h(s))])\chi(y, [0, h(s)]) \, ds \, dy \, dx
\leq \int_0^\infty \int_0^{k^{-1}(x)} h^{-1}(y) \, dy \, dx
\leq \left( \int_0^1 [h^{-1}(y)]^2 \, dy \right)^{1/2} \left( \int_0^1 [k^{-1}(y)]^2 \, dy \right)^{1/2}
= 2 \left[ \int_0^\infty sh(s) \, ds \right]^{1/2} \left[ \int_0^\infty rk(r) \, dr \right]^{1/2},
\]
where the last equality can be verified by comparing two methods for computing the volume of a solid of revolution. Now interchanging $h$ and $k$ completes the proof of the lemma. \qed

**Lemma 4.** There is a positive number $C$ such that if $E$ and $F$ are measurable subsets of $\mathbb{R}^2$, then
\[
\int_{\mathbb{R}^2} \chi(x, E)S\chi(\cdot, F)(x) \, dx \leq C|E|^{1/2}|F|^{1/2}.
\]

**Proof.** The operator $S$ is given by the formula
\[
Sg(re^{i\theta}) = \frac{1}{r} \int_{-\infty}^\infty g(re^{i\theta} + te^{i(\theta + \xi)}) \, dt,
\]
so we will show that
\[
\int_0^\infty \int_{-\infty}^\infty \int_0^{2\pi} \chi(re^{i\theta}, E)\chi(re^{i\theta} + te^{i(\theta + \xi)}, F) \, d\theta \, dt \, dr \leq C|E|^{1/2}|F|^{1/2}.
\]
We will actually consider only the integral $\int_0^\infty \int_{-\infty}^\infty \int_0^{2\pi}$—the integral $\int_0^\infty \int_{-\infty}^0 \int_0^{2\pi}$ is treated analogously. The change of variable
\[
x = x_r(\theta, t) = re^{i\theta} + te^{i(\theta + \xi)}
\]
shows that
\[
\int_0^\infty \int_0^\infty \int_0^{2\pi} \chi(re^{i\theta}, E)\chi(re^{i\theta} + te^{i(\theta + \frac{\pi}{2})}, F) \, d\theta \, dt \, dr
\]
\[
= \int_0^\infty \int_\{|x| > r\} \chi(p(r, x), E)\chi(x, F) \frac{dx}{\sqrt{|x|^2 - r^2}} \, dr,
\]
where \( p(r, x) \) is the point \( re^{i\theta} \) such that \( x \) can be written \( x = re^{i\theta} + te^{i(\theta + \frac{\pi}{2})} \) for some \( t > 0 \). We write the last integral as \( I_1 + I_2 \) where
\[
I_1 = \int_0^\infty \int_{\{4r > |x| > r\}} \chi(p(r, x), E)\chi(x, F) \frac{dx}{\sqrt{|x|^2 - r^2}} \, dr,
\]
and we begin by considering \( I_1 \):
\[
\int_0^\infty \int_{\{4r > |x| > r\}} \chi(p(r, x), E)\chi(x, F) \frac{dx}{\sqrt{|x|^2 - r^2}} \, dr
\]
\[
\leq C \int_0^\infty (r^{-\frac{1}{2}} |F \cap \{r < |x| < 4r\}|^{1/2})
\times (r^{\frac{1}{2}} \|\chi(p(r, x), E)(|x|^2 - r^2)^{-\frac{1}{2}} \chi(|x|, (r, \infty))\|^2_{L^{2,\infty}}) \, dr,
\]
since \( L^{2,\infty} \) is the dual of \( L^{2,1} \). Applying Hölder's inequality shows that this last integral is dominated by
\[
C \left( \int_0^\infty |F \cap \{r < |x| < 4r\}| \frac{dr}{r} \right)^{1/2}
\times \left( \int_0^\infty \|\chi(p(r, x), E)(|x|^2 - r^2)^{-\frac{1}{2}} \chi(|x|, (r, \infty))\|^2_{L^{2,\infty}} r \, dr \right)^{1/2}.
\]
The first term in parentheses is just \( (\log 4) \cdot |F| \). We will now prove that the second parenthesized integral is bounded by \( C|E| \) and hence that \( I_1 \leq C|E|^{1/2}|F|^{1/2} \). A sketch shows that if
\[
se^{i\phi} = re^{i\theta} + te^{i(\theta + \frac{\pi}{2})},
\]
then
\[
\exp(i\theta) = \exp(i[\phi - \cos^{-1}(r/s)]).
\]
Thus
\[
|\{\phi \in [0, 2\pi): p(r, se^{i\phi}) \in E\}| = |\{\phi \in [0, 2\pi): re^{i\phi} \in E\}|,
\]
and so, for \( \lambda > 0 \),
\[
|\{x: |x| > r, p(r, x) \in E, (|x|^2 - r^2)^{-\frac{1}{2}} > \lambda\}|
\]
\[
= \int_r^{\sqrt{r^2 + \lambda^{-2}}} |\{\phi \in [0, 2\pi): p(r, se^{i\phi}) \in E\}| s \, ds
\]
\[
= |\{\phi \in [0, 2\pi): re^{i\phi} \in E\}|/2\lambda^2.
\]
It follows that
\[ \int_0^\infty \| \chi(p(r, x), E)(|x|^2 - r^2)^{-\frac{1}{2}} \chi(|x|, (r, \infty)) \|_{L^2_r}^2 r dr \leq C|E| \]
as claimed. Thus the proof will be complete when we see that
\[ I_2 \leq C|E|^{1/2}|F|^{1/2}. \]
Now if \(|x| > 4r\), then
\[ \frac{1}{\sqrt{|x|^2 - r^2}} \leq \frac{2}{|x|}, \]
so
\[ \frac{I_2}{4\pi} \leq \int_0^\infty \int_0^\infty \frac{1}{2\pi} \int_0^{2\pi} \chi(p(r, se^{i\phi}), E)\chi(se^{i\phi}, F) d\phi ds dr \\
\leq \int_0^\infty \int_0^\infty \min \left\{ \frac{1}{2\pi} \int_0^{2\pi} \chi(p(r, se^{i\phi}), E) d\phi, \frac{1}{2\pi} \int_0^{2\pi} \chi(se^{i\phi}, F) d\phi \right\} ds dr. \]
As noted earlier,
\[ \int_0^{2\pi} \chi(p(r, se^{i\phi}), E) d\phi = \int_0^{2\pi} \chi(re^{i\phi}, E) d\phi. \]
Thus an application of Lemma 3 completes the proof. \(\Box\)

**References**


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