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PRINCIPAL EIGENVALUES FOR PROBLEMS WITH INDEFINITE WEIGHT FUNCTION ON R^N

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ABSTRACT. We investigate the existence of positive principal eigenvalues of the problem $-\Delta u(x) = \lambda g(x)u$ for $x \in R^n$; $u(x) \to 0$ as $x \to \infty$ where the weight function g changes sign on R^n. It is proved that such eigenvalues exist if g is negative and bounded away from 0 at ∞ or if $n \geq 3$ and $|g(x)|$ is sufficiently small at ∞ but do not exist if $n = 1$ or 2 and $\int_{R^n} g(x)\,dx > 0$.

1. INTRODUCTION

We shall discuss the existence of a principal eigenvalue for the linear elliptic problem

$$(1) \qquad -\Delta u(x) = \lambda g(x)u(x) \qquad \text{for } x \text{ in } R^n; \quad u(x) \to 0 \text{ as } |x| \to \infty,$$

where Δ denotes the Laplacian and $g\colon R^n \to R$ is a smooth function which changes sign on R^n, i.e. g is an indefinite weight function. By a principal eigenvalue of (1) we mean a value of λ corresponding to which there is a solution u of (1) with $u(x) > 0$ for $x \in R^n$; in this case u is termed a principal eigenfunction. Principal eigenvalues for similar problems on bounded regions with Dirichlet and Neumann boundary conditions have been discussed by various authors (see e.g. Bocher [5], Hess and Kato [9], Brown and Lin [2], Flecking and Lapidus [6]). The study of such problems has been motivated in part by the wish to understand related nonlinear boundary value problems such as

$$(2) \qquad -\Delta u(x) = \lambda g(x)f(u(x)) \qquad \text{for } x \text{ in } \Omega$$

where, e.g. $f(u) = u(1-u)$ (see Fleming [7]). A branch of positive solutions may bifurcate from the zero solution of (2) precisely when λ is a principal

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eigenvalue of the corresponding linearized problem. Such a nonlinear problem on all of R^n was studied by Brown *et al.* in [3] and [4] and some of the results of the present paper were motivated by the results therein.

In §2 we prove the existence of a positive principal eigenvalue provided g is negative and bounded away from 0 at ∞ and so generalize the result of [4] where it is shown that if g is, in addition, radially symmetric then bifurcation of positive solutions occurs at some positive value of λ. Our results complement the results of Bonnet in [1] where the situation $n = 2$ is discussed. In §3 we show that, if $n = 1$ or 2 and $\int_{R^n} g\,dx > 0$, then there is no positive principal eigenvalue. Finally in §4 we show that, if $n \geq 3$ and $g(x)$ decays faster than $1/|x|^2$, then there exists a positive principal eigenvalue. The dependence of the nature of the spectrum on the spatial dimension appears elsewhere in the literature on the spectral theory of linear operators on $L_2(R^n)$ (e.g. Cwikel-Lieb-Rosenbljum bound, see Reed and Simon [11]). In our case a crucial feature of the argument in §4 is the use of the Hardy inequality which holds only when $n \geq 3$.

2. Existence of principal eigenvalue when g is negative at ∞

We shall establish the following result.

Theorem 2.1. *Suppose that there exist $R > 0$ and $K > 0$ such that $g(x) \leq -K$ for $|x| > R$. Then there exists a positive principal eigenvalue of* (1) *with corresponding principal eigenfunction lying in $L_2(R^n)$.*

First we prove a preliminary lemma.

Lemma 2.2. *Suppose g satisfies the hypotheses of Theorem* 2.1. *Then*

$$\lambda_1 = \inf\left\{\int_{R^n} |\nabla u|^2\,dx \Big/ \int_{R^n} gu^2\,dx : u \in H^1(R^n),\ \int_{R^n} gu^2\,dx > 0\right\} > 0.$$

Proof. Let B denote a ball such that $\int_B g\,dx < 0$ and $g(x) < 0$ whenever $x \notin B$. It can be shown (see Brown, Lin, and Tertikas [3]) that there exists $c_1 > 0$ such that

$$\int_B |\nabla u|^2\,dx \geq c_1 \int_B u^2\,dx$$

for all $u \in H^1(B)$ with $\int_B gu^2\,dx > 0$.

Suppose $u \in H^1(R^n)$ and $\int_{R^n} gu^2\,dx > 0$. Then $\int_B gu^2\,dx > 0$ and

$$\int_{R^n} |\nabla u|^2\,dx \geq \int_B |\nabla u|^2\,dx \geq c_1 \int_B u^2\,dx \geq c_1c_2 \int_B gu^2\,dx \geq c_1c_2 \int_{R^n} gu^2\,dx$$

where $c_2 > 0$ is chosen such that $c_2 g(x) \leq 1$ for all $x \in B$. Hence $\{\int_{R^n} |\nabla u|^2\,dx / \int_{R^n} gu^2\,dx : u \in H^1(R^n),\ \int_{R^n} gu^2\,dx > 0\}$ is bounded below by c_1c_2 and so the proof is complete.

Proof of Theorem 2.1. We prove that λ_1 as defined in Lemma 2.2 is the required principal eigenvalue.

Let L denote the self-adjoint operator on $L_2(R^n)$ generated by the differential expression $-\Delta-\lambda_1 g$. Since there exists $K_1 > 0$ such that $-\lambda_1 g(x)+K_1 \geq 0$ for all x, the symmetric operator defined by $-\Delta u - \lambda_1 g$ on $C_0^\infty(R^n)$ is essentially self-adjoint (see Reed and Simon [11], Theorem X.28) and so has closure L. It follows that the domain of L, $D(L)$, is contained in $H^1(R^n)$ and that

$$(Lu, v) = \int_{R^n} (\nabla u \cdot \nabla v - \lambda_1 guv)\, dx \qquad \text{for all } u, v \in D(L)$$

where (,) denotes the usual inner product on $L_2(R^n)$. Because of the definition of λ_1, there exists a sequence of functions $\{u_k\}$ in $H^1(R^n)$ such that $\int_{R^n} u_k^2\, dx = 1$, $\int_{R^n} gu_k^2\, dx > 0$ and $\int_{R^n} |\nabla u_k|^2\, dx / \int_{R^n} gu_k^2\, dx \to \lambda_1$. Since $C_0^\infty(R^n)$ is dense in $H^1(R^n)$, we may assume without loss of generality that $\{u_k\} \subset C_0^\infty(R^n)$. Then

$$\lim_{k\to\infty} (Lu_k, u_k) = \lim_{k\to\infty} \left[\int_{R^n} gu_k^2\, dx\right] \left[\int_{R^n} |\nabla u_k|^2\, dx \Big/ \int_{R^n} gu_k^2\, dx - \lambda_1\right] = 0$$

as $\int_{R^n} gu_k^2\, dx \leq \max g(x)$ for all k. It is also easy to see from the definition of λ_1 that $(Lu, u) \geq 0$ for all $u \in D(L)$. Thus $\inf\{(Lu, u) : u \in D(L), (u, u) = 1\} = 0$ and so $\inf \sigma(L) = 0$ where $\sigma(L)$ denotes the spectrum of L.

Let $\hat{g}$ be a smooth function such that $\hat{g}(x) = g(x)$ when $|x| > R$ and $\hat{g}(x) \leq -K$ for all x. Let $\hat{L}$ denote the self-adjoint operator generated by $-\Delta - \lambda_1 \hat{g}$. Since

$$(\hat{L}u, u) = \int_{R^n} (|\nabla u|^2 - \lambda_1 \hat{g} u^2)\, dx \geq \lambda_1 K \int_{R^n} u^2\, dx \qquad \text{for all } u \in D(\hat{L})$$

it follows that the spectrum and so the essential spectrum of $\hat{L}$ is contained in $[\lambda_1 K, \infty)$. Moreover $L = \hat{L} - \lambda_1(g - \hat{g})$ where $g - \hat{g}$ has compact support and so the essential spectrum of L coincides with the essential spectrum of $\hat{L}$ (see Reed and Simon [12], §XIII.4). Thus 0 lies in $\sigma(L)$ but not in the essential spectrum of L and so must be an eigenvalue of finite multiplicity of L, i.e. there exists nonzero $u \in L_2(R^n)$ such that $Lu = 0$.

This eigenfunction u is a weak solution of $-\Delta u = \lambda_1 gu$ on R^n and so Weyl's lemma (see Reed and Simon [11], p. 53) shows that u is also a classical solution. Moreover as $\inf \sigma(L)$ is an eigenvalue of L, it follows from Reed and Simon [12], Theorem XIII.48 that the corresponding eigenfunction is strictly positive.

We now obtain bounds for λ_1 in terms of eigenvalues for equations on bounded regions. Let B again denote a ball such that $\int_B g\, dx < 0$ and $g(x) < 0$ whenever $x \notin B$. Let $\delta_1(B)$ and $\nu_1(B)$ denote the positive principal eigenvalues of

$$-\Delta u(x) = \lambda g(x) u(x) \qquad \text{for } x \text{ in } B$$

with Dirichlet and Neumann boundary conditions, respectively. It can be shown that (see Brown and Lin [2] or Manes and Micheletti [10])

$$\delta_1(B) = \inf\left\{I(u, B) : u \in H_0^1(B), \int_B gu^2\, dx > 0\right\}$$

and

$$\nu_1(B) = \inf\left\{I(u, B) : u \in H^1(B), \int_B gu^2\,dx > 0\right\},$$

where $I(u, \Omega) = \int_\Omega |\nabla u|^2\,dx / \int_\Omega gu^2\,dx$. Clearly $\nu_1(B) < \delta_1(B)$ and as can be seen from the Proof of Lemma 2.2 the condition $\int_B g\,dx < 0$ ensures that $\nu_1(B) > 0$. We have shown above that

$$\lambda_1 = \inf\left\{I(u, R^n) : u \in H^1(R^n), \int_{R^n} gu^2\,dx > 0\right\}.$$

Theorem 2.3. $\nu_1(B) \le \lambda_1 \le \delta_1(B)$.

Proof. Let $u \in H^1(R^n)$ with $\int_{R^n} gu^2\,dx > 0$. Then $u|_B \in H^1(B)$ and, since $g < 0$ on R^n/B, $\int_B gu^2\,dx > 0$. Hence $\nu_1(B) \le \lambda_1$.

Let $u \in H_0^1(B)$ with $\int_B gu^2\,dx > 0$. If u is defined to be zero outside of B then $u \in H^1(R^n)$ and $\int_{R^n} gu^2\,dx > 0$. Hence $\lambda_1 \le \delta_1(B)$.

3. Nonexistence of a positive principal eigenvalue

Throughout this section we assume that $\int_{R^n} g(x)dx > 0$. We allow the case where $\int_{R^n} g(x) = +\infty$ but our assumption implies that $\int_{R^n} g^-(x)dx < \infty$ where $g^- = \max\{-g, 0\}$. We first extend a result of [3].

Lemma 3.1. *Suppose $n = 1$ or 2 and that $\int_{R^n} g(x)dx > 0$. If $\delta_1(r)$ denotes the positive principal eigenvalue of the problem*

$$-\Delta u(x) = \lambda g(x)u(x) \quad \textit{for } |x| \le r; \qquad u(x) = 0 \quad \textit{for } |x| = r$$

then $\lim_{r\to\infty} \delta_1(r) = 0$.

Proof. We consider only the case where $n = 2$; the case $n = 1$ is similar but simpler. Let $M = \min\{1, \frac{1}{2}\int_{R^2} g(x)\,dx\}$. Choose $R > 0$ such that

$$\int_{|x|\le R} g(x)dx \ge M; \qquad \int_{|x|\ge R} g^-(x)dx \le \tfrac{1}{2}M.$$

Let $\varepsilon > 0$. Define a continuous radially symmetric function v as follows

$$\begin{aligned} v(r) &= 1 && \text{if } r \le R, \\ v'(r) &= -\varepsilon/r && \text{if } R \le r \le Z, \\ v(r) &= 0 && \text{if } r \le Z. \end{aligned}$$

Then v is a decreasing function for $R \le r \le Z$ with $v(R) = 1$ and $v(Z) = 0$; clearly Z is a function of R and ε. Since $v(r) = -\varepsilon \ln r + b$ for some positive constant b on $[R, Z]$, we must have that

$$-\varepsilon \ln R + b = 1; \qquad -\varepsilon \ln Z + b = 0$$

and so

$$\varepsilon(\ln Z - \ln R) = 1.$$

If $r > Z$

$$\int_{|x|\le r} |\nabla v|^2\, dx = \int_0^r r v_r^2\, dr = \int_R^Z \varepsilon^2/r\, dr = \varepsilon^2(\ln Z - \ln R) = \varepsilon.$$

Moreover

$$\int_{|x|\le r} g(x) v^2\, dx = \int_{|x|\le R} g(x)\, dx + \int_{R\le|x|\le r} g(x) v^2\, dx$$
$$\ge M - \int_{R\le|x|\le r} g^-(x)\, dx \ge \frac{1}{2} M.$$

Hence, if $r > Z$,

$$\delta_1(r) = \inf\left\{ I(u, B_r) : u \in H_0^1(B_r),\ \int_{B_r} g u^2\, dx > 0 \right\}$$
$$\le I(v, B_r) \le 2\varepsilon/M,$$

where I denotes the Rayleigh quotient as in the previous section and $B_r = \{x \in R^2 : |x| \le r\}$. This completes the proof.

Theorem 3.2. *Suppose* $n = 1$ *or* 2 *and* $\int_{R^n} g\, dx > 0$. *Then there does not exist a positive principal eigenvalue of* (1).

Proof. Suppose that the theorem is false and let λ and u denote a principal eigenvalue and corresponding principal eigenfunction.

It is easy to see that by choosing B to be a sufficiently small ball on which $g > 0$ we can make $\delta_1(B)$ as large as we please. Hence, as $\lim_{r\to\infty} \delta_1(B_r) = 0$ and the principal eigenvalue of the Dirichlet problem depends continuously on the domain, we can find a ball B such that $\delta_1(B) = \lambda$. If φ denotes a corresponding principal eigenfunction, then

$$(3) \qquad -\Delta\varphi = \lambda g \varphi \quad \text{on } B; \qquad \varphi|_{\partial B} = 0.$$

Choose $K > \lambda \sup\{|g(x)| : x \in B\}$. Then

$$-\Delta\varphi + (K - \lambda g)\varphi = K\varphi \ge 0 \qquad \text{on } B$$

and it follows from the maximum principle that $\partial\varphi/\partial n < 0$ on ∂B.

Since u is a principal eigenfunction on R^n,

$$(4) \qquad -\Delta u = \lambda g u \quad \text{on } B; \qquad u|_{\partial B} > 0.$$

Multiplying equation (3) by u and equation (4) by φ, subtracting and integrating we obtain $\int_{\partial B} u(\partial\varphi/\partial n)\, dS = 0$ which is impossible and so the proof is complete.

4. Existence of principal eigenvalue when g is small at infinity

In this section we shall assume that $|g(x)| \le K(1+|x|^2)^{-\alpha}$ for some constants $K > 0$ and $\alpha > 1$. This smallness assumption is compatible with the positivity assumption of the previous section, viz. $\int_{R^n} g\, dx > 0$; however the nature of the result changes as we now consider only the case $n \geq 3$. Using these assumptions we shall prove the existence of a principal eigenvalue.

We establish our results by following the variational approach of Weinberger [13]. Define the Hilbert space

$$V = \left\{ u\colon R^n \to R\colon \int_{R^n} (|\nabla u|^2 + (1+|x|^2)^{-1} u^2)\, dx < \infty \right\}$$

with inner product

$$(u, v) = \int_{R^n} (\nabla u . \nabla v + (1+|x|^2)^{-1} uv) dx$$

and define sesquilinear forms $a, b\colon V \times V \to R$ by

$$a(u, v) = \int_{R^n} \nabla u . \nabla v\, dx\, ;\ b(u, v) = \int_{R^n} guv\, dx .$$

Clearly $a(u, u) \leq (u, u)$. By Hardy's inequality there exists a constant k such that

$$\int_{R^n} |\nabla u|^2 \geq k \int_{R^n} u^2/|x|^2\, dx \qquad \text{for all } u \in C_0^\infty(R^n).$$

Hence

$$a(u, u) = \int_{R^n} |\nabla u|^2\, dx \geq \frac{1}{2} \int_{R^n} (|\nabla u|^2 + ku^2/|x|^2) dx \geq \frac{1}{2} k_1 (u, u),$$

where $k_1 = \min\{1, k\}$ for all $u \in C_0^\infty(R^n)$. Since $C_0^\infty(R^n)$ is dense in V, it follows that $a(u, u) \geq \frac{1}{2} k_1 (u, u)$ for all u in V. Thus $a(\cdot, \cdot)$ is an equivalent inner product for V. Henceforth we shall assume that the inner product for V is that induced by a and denote the corresponding norm by $\| \ \|_v$.

For all $u, v \in V$

$$\begin{aligned} |b(u, v)| &= \left| \int_{R^n} guv\, dx \right| \\ &\leq K \left\{ \int_{R^n} (1+|x|^2)^{-\alpha} u^2\, dx \right\}^{1/2} \left\{ \int_{R^n} (1+|x|^2)^{-\alpha} v^2\, dx \right\}^{1/2} \\ &\leq K(u, u)^{1/2} (v, v)^{1/2} \leq K_1 \|u\|_v \|v\|_v \qquad \text{for some constant } K_1 . \end{aligned}$$

Hence by the Riesz representation theorem we can define a linear operator $T\colon V \to V$ such that

$$b(u, v) = a(Tu, v) \qquad \text{for all } u, v \in V .$$

It is straightforward to check that T is bounded and self-adjoint.

We now show that T is a compact operator. Suppose that $\{u_k\}$ is a bounded sequence in V. Then for all positive integers k, l

$$\begin{aligned}\|Tu_k - Tu_l\|_v^2 &= a(T(u_k - u_l), T(u_k - u_l)) = b(u_k - u_l, T(u_k - u_l)) \\ &= \int_{R^n} g(u_k - u_l)T(u_k - u_l)\,dx \\ &\le K\left\{\int_{R^n}(1+|x|^2)^{-\alpha}(u_k - u_l)^2\,dx\right\}^{l/2} \\ &\quad \times \left\{\int_{R^n}(1+|x|^2)^{-\alpha}(Tu_k - Tu_l)^2\,dx\right\}^{1/2} \\ &\le L\left\{\int_{R^n}(1+|x|^2)^{-\alpha}(u_k - u_l)^2\,dx\right\}^{1/2}\|Tu_k - Tu_l\|v\end{aligned}$$

for some constant L.

Thus

$$\|Tu_k - Tu_l\|_v^2 \le L^2\left\{\int_{R^n}(1+|x|^2)^{-\alpha}(u_k - u_l)^2\,dx\right\}. \tag{5}$$

We use (5) to produce a convergent subsequence of $\{Tu_k\}$. Since $\{u_k\}$ is bounded in V, $\{u_k\}$ is bounded in $H^1(B)$ for every ball B in R^n and so has a convergent subsequence in $L_2(B)$. Thus by using a diagonalization procedure we can find a subsequence, for convenience again denoted by $\{u_k\}$ which converges in $L_2(B)$ for every bounded subset B of R^n. We now show that the sequence $\{Tu_k\}$ is Cauchy.

Let $\varepsilon > 0$. Since $\{u_k\}$ is bounded in V, there exists a constant $M > 0$ such that $\int_{R^n}(1+|x|^2)^{-1}u_k^2\,dx < M$ for all k. Clearly $\int_{R^n}(1+|x|^2)^{-1}(u_k - u_1)^2\,dx \le 4M$ for all k, l. Choose $R > 0$ such that $4M(1+R^2)^{1-\alpha} < \varepsilon$. Because $\{u_k\}$ is Cauchy on $L_2(B_R)$, there exists a positive integer N such that $\int_{B_R}(u_k - u_1)^2\,dx < \varepsilon$ if k, $l > N$. Hence by (5)

$$\begin{aligned}&\|Tu_k - Tu_l\|_v^2 \\ &\quad\le L^2\left\{\int_{B_R}(1+|x|^2)^{-\alpha}(u_k - u_l)^2\,dx + \int_{|x|\ge R}(1+|x|^2)^{-\alpha}(u_k - u_l)^2\,dx\right\} \\ &\quad\le L^2\left\{\varepsilon + (1+R^2)^{1-\alpha}\int_{|x|\ge R}(1+|x|^2)^{-1}(u_k - u_l)^2\,dx\right\} \qquad \text{if } k, l > N \\ &\quad\le L^2\left\{\varepsilon + 4M(1+R^2)^{1-\alpha}\right\} < 2L^2\varepsilon.\end{aligned}$$

Hence $\{Tu_k\}$ is Cauchy and so T is a compact operator.

Since T is compact, the largest eigenvalue μ_l of T is given by $\mu_1 = \sup_{u\in V} a(Tu, u)/a(u, u)$, i.e.

$$\mu_1 = \sup_{u\in V} b(u, u)/a(u, u) = \sup_{u\in V}\int_{R^n} gu^2\,dx \Big/ \int_{R^n}|\nabla u|^2\,dx.$$

If g is positive on an open set G of R^n, there exists $u \in V$ with support in G and so with $\int_{R^n} gu^2\,dx / \int_R |\nabla u|^2\,dx > 0$. Hence $\mu_1 > 0$.

Let φ denote an eigenfunction corresponding to the eigenvalue μ_1 of T. Since $T\varphi = \mu_1\varphi$,

$$b(\varphi, v) = a(T\varphi, v) = \mu_1 a(\varphi, v) \qquad \text{for all } v \in V,$$

i.e.

$$\mu_1 \int \nabla\varphi\cdot\nabla v\,dx = \int_{R^n} g\varphi v\,dx \qquad \text{for all } v \in V.$$

Thus φ is a weak solution of $-\Delta u = \mu_1^{-1} gu$ and so by Weyl's lemma is a classical solution.

We now prove that φ does not change sign on R^n. Let φ^+ and φ^- denote the positive and negative parts of φ, i.e. $\varphi = \varphi^+ + \varphi^-$. Then φ^+, $\varphi^- \in V$ and

$$a(\varphi, \varphi) = a(\varphi^+, \varphi^+) + a(\varphi^-, \varphi^-); \qquad b(\varphi, \varphi) = b(\varphi^+, \varphi^+) + b(\varphi^-, \varphi^-).$$

Since

$$\frac{b(\varphi^+, \varphi^+)}{a(\varphi^+, \varphi^+)} - \frac{b(\varphi, \varphi)}{a(\varphi, \varphi)} = \frac{b(\varphi^+, \varphi^+)a(\varphi^-, \varphi^-) - a(\varphi^+, \varphi^+)b(\varphi^-, \varphi^-)}{a(\varphi^+, \varphi^+)a(\varphi, \varphi)}$$

and

$$\frac{b(\varphi^-, \varphi^-)}{a(\varphi^-, \varphi^-)} - \frac{b(\varphi, \varphi)}{a(\varphi, \varphi)} = \frac{a(\varphi^+, \varphi^+)b(\varphi^-, \varphi^-) - a(\varphi^-, \varphi^-)b(\varphi^+, \varphi^+)}{a(\varphi^-, \varphi^-)a(\varphi, \varphi)}$$

it follows that

$$\max\left\{\frac{b(\varphi^+, \varphi^+)}{a(\varphi^+, \varphi^+)}, \frac{b(\varphi^-, \varphi^-)}{a(\varphi^-, \varphi^-)}\right\} \geq \frac{b(\varphi, \varphi)}{a(\varphi, \varphi)} = \mu_1.$$

Hence, because of the supremum definition of μ_1, we must have that either

$$\frac{b(\varphi^+, \varphi^+)}{a(\varphi^+, \varphi^+)} = \mu_1 \qquad \text{or} \qquad \frac{b(\varphi^-, \varphi^-)}{a(\varphi^-, \varphi^-)} = \mu_1.$$

Hence by Theorem 5.2 of Weinberger [13] it follows that either φ^+ or φ^- is an eigenfunction corresponding to μ_1 i.e. there exists an eigenfunction corresponding to μ_1 which does not change sign on R^n.

For convenience we again denote this eigenfunction by φ and suppose that $\varphi \geq 0$ on R^n. Suppose that there exists $x_0 \in R^n$ such that $\varphi(x_0) = 0$. Let B denote any ball in R^n centered at x_0 such that φ is not identically equal to 0 on B. Choose $C > 0$ such that $C - \mu_1^{-1} g(x) > 0$ for all x in B. Then

$$-\Delta\varphi + (C - \mu_1^{-1} g(x))\varphi \geq C\varphi \geq 0 \quad \text{on } B; \qquad \varphi \geq 0 \quad \text{on } \partial B$$

and so by the maximum principle $\varphi > 0$ on B. Hence $\varphi(x) > 0$ for all $x \in R^n$. (The argument used in this paragraph is similar to an argument used in Gossez and Lami Dozo in [8].)

Thus we have proved

Theorem 4.1. *Suppose* $n \geq 3$ *and there exists* $K > 0$ *and* $\alpha > 1$ *such that* $|g(x)| \leq K(1+|x|^2)^{-\alpha}$. *Then there exists a positive principal eigenvalue of* (1).

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