COMMUTING AND CENTRALIZING MAPPINGS IN PRIME RINGS

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Dedicated to the memory of my father

Abstract. Let $R$ be a ring. A mapping $F: R \to R$ is said to be commuting on $R$ if $[F(x), x] = 0$ holds for all $x \in R$. The main purpose of this paper is to prove the following result, which generalizes a classical result of E. Posner: Let $R$ be a prime ring of characteristic not two. Suppose there exists a nonzero derivation $D: R \to R$, such that the mapping $x \mapsto [D(x), x]$ is commuting on $R$. In this case $R$ is commutative.

Preliminaries

Throughout this paper $R$ will represent an associative ring with center $Z(R)$. We write $[x, y]$ for $xy - yx$, and use the identities $[xy, z] = [x, z]y + x[y, z]$, $[x, yz] = [x, y]z + y[x, z]$. Recall that $R$ is prime if $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping $D$ from $R$ to $R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. An additive mapping $D$ from $R$ to $R$ is called a Jordan derivation if $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. Obviously, every derivation is a Jordan derivation. The converse is in general not true. A well-known result first proved by I. N. Herstein [5] states that every Jordan derivation on a prime ring of characteristic not two is a derivation. A brief proof of Herstein's result can be found in [2]. A derivation $D$ is inner if there exists $a \in R$, such that $D(x) = [a, x]$ holds all for all $x \in R$. A mapping $F$ from $R$ to $R$ is said to be commuting on $R$ if $[F(x), x] = 0$ for all $x \in R$, and is said to be centralizing on $R$ if $[F(x), x] \in Z(R)$ holds for all $x \in R$. There has been considerable interest in commuting, centralizing, and related mappings in prime and semiprime rings (see [1, 3, 4, 6, 7, 8, and 10] where further references can be found). Our methods are somewhat different from those employed by other authors.
The results

We shall need the following well-known and frequently used lemmas.

**Lemma 1** [9, Lemma 1]. Let $D: R \to R$ be a derivation, where $R$ is a prime ring. Suppose that either

(i) $aD(x) = 0$, \quad $x \in R$

or

(ii) $D(x)a = 0$, \quad $x \in R$

holds. In both cases we have $a = 0$ or $D = 0$.

**Lemma 2** [9, Lemma 3]. Let $D: R \to R$ be a nonzero derivation, where $R$ is a prime ring. Suppose that $D$ is commuting on $R$. In this case $R$ is commutative.

We shall start our investigations with our main result.

**Theorem 1.** Let $R$ be a noncommutative prime ring of characteristic not two. Suppose there exists a derivation $D: R \to R$, such that the mapping $x \mapsto [D(x), x]$ is commuting on $R$. In this case $D = 0$.

A classical result in the theory of centralizing mappings is a theorem of E. Posner [9, Theorem 2] which states that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative (obviously, Lemma 2 is a special case of this result). Neglecting the fact that in the above result we have an additional assumption concerning the characteristic of the ring, we can say, that Theorem 1 generalizes Posner's theorem.

**Proof of Theorem 1.** We have

$$[[D(x), x], x] = 0, \quad x \in R.$$  \hspace{1cm} (1)

Let us introduce a mapping $B(\cdot, \cdot): R \times R \to R$ by the relation

$$B(x, y) = [D(x), y] + [D(y), x], \quad x, y \in R.$$  \hspace{1cm} (2)

It is obvious that $B(\cdot, \cdot)$ is symmetric (i.e. $B(x, y) = B(y, x)$ for all $x, y \in R$) and additive in both arguments. Moreover, a simple calculation shows that the relation

$$B(xy, z) = B(x, z)y + xB(y, z) + D(x)[y, z] + [x, z]D(y)$$

is fulfilled for all $x, y, z \in R$. We introduce also a mapping $f$ from $R$ to $R$ by $f(x) = B(x, x)$. We have

$$f(x) = 2[D(x), x], \quad x \in R.$$  \hspace{1cm} (3)

Obviously, the mapping $f$ satisfies the relation

$$f(x + y) = f(x) + f(y) + 2B(x, y), \quad x, y \in R.$$  \hspace{1cm} (4)

Throughout the proof we shall use the mapping $B(\cdot, \cdot)$ and the relations (2), (3), and (4) without specific reference. The relation (1) can now be written in the form

$$[f(x), x] = 0, \quad x \in R.$$  \hspace{1cm} (5)
The linearization of (5) gives

\[ (f(x), y) + [f(y), x] + 2[B(x, y), x] + 2[B(x, y), y] = 0, \quad x, y \in R. \]

The substitution \(-x\) for \(x\) in the above relation leads to

\[ (f(x), y) - [f(y), x] + 2[B(x, y), x] - 2[B(x, y), y] = 0, \quad x, y \in R. \]

From (6) and (7) we obtain

\[ (f(x), y) + 2[B(x, y), x] = 0, \quad x, y \in R. \]

Let us replace in (8) \(y\) by \(xy\). Then

\[ 0 = [f(x), xy] + 2[B(xy, x), x] = [f(x), xy] + 2[f(x)y + xB(x, y) + D(x)[y, x], x] + 2[D(x), x][y, x] + 2D(x)[y, x], x] = 0. \]

Using in the above calculation (5) and (8) we arrive at

\[ 3f(x)[y, x] + 2D(x)[y, x], x] = 0, \quad x, y \in R. \]

Similarly, we obtain the relation

\[ 3[y, x]f(x) + 2[[y, x], x]D(x) = 0, \quad x, y \in R \]

putting in (8) \(yx\) instead of \(y\). We intend to prove that

\[ 3f(x)D(x) - D(x)f(x) = 0, \quad x \in R \]

holds. For this purpose we write \(yz\) instead of \(y\) in (9). We have

\[ 0 = 3f(x)[yz, x] + 2D(x)[yz, x], x] = 3f(x)[y, x]z + 3f(x)y[z, x] + 2D(x)[y, x], x]z + 4D(x)[y, x][z, x] + 2D(x)y[[z, x], x]. \]

By (9) the above calculation reduces to

\[ 3f(x)y[z, x] + 4D(x)[y, x][z, x] + 2D(x)y[[z, x], x] = 0, \quad x, y, z \in R. \]

Putting in the above relation \(y = D(x)\) we obtain

\[ 3f(x)D(x)[z, x] + 2D(x)x + f(x)[z, x] + 2D(x)^2[[z, x], x] = 0, \quad x, z \in R, \]

which yields

\[ 3f(x)D(x)[z, x] = 0, \quad x, z \in R \]

according to (9). In other words we have proved the relation

\[ (3f(x)D(x) - D(x)f(x))[y, x] = 0, \quad x, y \in R. \]

Now we are ready for the proof of (11). There is nothing to prove if \(x \in Z(R)\), since in this case \(f(x) = 0\). Hence we can restrict our attention on the case \(x \notin Z(R)\). In this case \(y \mapsto [x, y]\) is a nonzero inner derivation, which means that from (12) and Lemma 1 it follows \(3f(x)D(x) - D(x)f(x) = 0\). Thus the relation (11) is proved. Similarly one proves the relation

\[ 3D(x)f(x) - f(x)D(x) = 0, \quad x \in R \]

Finally, we prove (12). For this purpose we write \(z\) instead of \(y\) in (11). We have

\[ 0 = 3f(x)[z, x] + 2D(x)[z, x], x] = 3f(x)[y, x]z + 3f(x)y[z, x] + 2D(x)[y, x], x]z + 4D(x)[y, x][z, x] + 2D(x)y[[z, x], x]. \]

By (9) the above calculation reduces to

\[ 3f(x)y[z, x] + 4D(x)[y, x][z, x] + 2D(x)y[[z, x], x] = 0, \quad x, y, z \in R. \]

Putting in the above relation \(y = D(x)\) we obtain

\[ 3f(x)D(x)[z, x] + 2D(x)x + f(x)[z, x] + 2D(x)^2[[z, x], x] = 0, \quad x, z \in R, \]

which yields

\[ 3f(x)D(x)[z, x] = 0, \quad x, z \in R \]

according to (9). In other words we have proved the relation

\[ (3f(x)D(x) - D(x)f(x))[y, x] = 0, \quad x, y \in R. \]

Now we are ready for the proof of (11). There is nothing to prove if \(x \in Z(R)\), since in this case \(f(x) = 0\). Hence we can restrict our attention on the case \(x \notin Z(R)\). In this case \(y \mapsto [x, y]\) is a nonzero inner derivation, which means that from (12) and Lemma 1 it follows \(3f(x)D(x) - D(x)f(x) = 0\). Thus the relation (11) is proved. Similarly one proves the relation

\[ 3D(x)f(x) - f(x)D(x) = 0, \quad x \in R \]
starting from (10). From (11) and (13) one obtains easily that

\[ D(x)f(x) = f(x)D(x) = 0, \quad x \in R \]

holds. The linearization of the relation \( D(x)f(x) = 0 \) gives

\[
D(x)\left(f(x) + f(y) + 2B(x, y)\right) = D(x)f(x) + D(y)f(x) + D(x)f(y) + D(y)f(y) + 2D(x)B(x, y) + 2D(y)B(x, y)
\]

which reduces to

\[ D(x)f(y) + D(y)f(x) + 2D(x)B(x, y) + 2D(y)B(x, y) = 0, \quad x, y \in R. \]

The substitution \(-x\) for \(x\) in (15) gives

\[ -D(x)f(y) + D(y)f(x) + 2D(x)B(x, y) - 2D(y)B(x, y) = 0, \quad x, y \in R. \]

Combining (15) with (16) we arrive at

\[ D(y)f(x) + 2D(x)B(x, y) = 0, \quad x, y \in R. \]

Put in (17) \(yx\) for \(y\). Then

\[
D(y)\left(f(x) + 2D(x)B(yx, x)\right) = D(y)f(x) + yD(x)f(x) + 2D(x)B(y, x)x + 2D(x)yf(x) + 2D(x)yB(x, y)
\]

which leads to

\[ D(y)f(x) + 2D(x)B(x, y)x + 2D(y)f(x) + 2D(x)yf(x) + 2D(x)yB(x, y) = 0, \quad x, y \in R
\]

according to (14). The relation (17) makes us possible to write \(-D(y)f(x)\) instead of \(2D(x)B(x, y)\) in the above relation. Thus we have

\[
D(y)[x, f(x)] + 2D(x)yf(x) + 2D(x)[y, x]D(x) = 0,
\]

which yields

\[ D(x)yf(x) + D(x)[y, x]D(x) = 0, \quad x, y \in R
\]

according to (5). Let us write in (18) \(yx\) for \(y\). Then

\[
D(x)\left(xyf(x) + D(x)[xy, x]D(x)\right) = D(x)xyf(x) + D(x)[xy, x]D(x) = D(x)x[y, x]D(x).
\]

Thus we have

\[ D(x)xyf(x) + D(x)x[y, x]D(x) = 0, \quad x, y \in R.
\]

Left multiplication of the relation (18) by \(x\) gives

\[ xD(x)xf(x) + xD(x)[y, x]D(x) = 0, \quad x, y \in R.
\]

Combining (19) with (20) we arrive at

\[ f(x)yf(x) + f(x)[y, x]D(x) = 0, \quad x, y \in R.
\]

Our next step is to prove the relation

\[ 3f(x)yf(x) + 4f(x)[y, x]D(x) = 0, \quad x, y \in R.
\]

For this purpose we write in (10) \(yz\) instead of \(y\). We have

\[
0 = 3[yz, x]f(x) + 2[[yz, x], x]D(x) = 3[y, x]z f(x) + 3[y, x]z f(x) + 2[[y, x], x]zD(x) + 4[y, x]zD(x) + 4[y, x]zD(x)
\]

which leads to

\[ 3[y, x]z f(x) + 2[[y, x], x]zD(x) + 4[y, x]zD(x) + 4[y, x]zD(x) = 0, \quad x, y, z \in R
\]
according to (10). Putting in (23) \( y = 2D(x) \) and making use of (5) we arrive at
\[ 3f(x)zf(x) + 4f(x)[z, x]D(x) = 0, \quad x, z \in R \]
which completes the proof of (22). From (21) and (22) one obtains immediately
\[ f(x)yf(x) = 0, \quad x, y \in R \]
which implies \( f(x) = 0, \quad x \in R \) by primeness of \( R \). Thus we have proved that \([D(x), x] = 0\) holds for all \( x \in R \), which yields \( D = 0 \) by Lemma 2. The proof of the theorem is complete.

We are ready for our next result.

**Theorem 2.** Let \( R \) be a noncommutative prime ring of characteristic different from two and three. Suppose there exists a derivation \( D: R \to R \), such that the mapping \( x \mapsto [D(x), x] \) is centralizing on \( R \). In this case \( D = 0 \).

**Proof.** Throughout the proof we shall use the same notation as in the proof of Theorem 1. The assumption of the theorem can be written as follows
\[ (24) \quad [f(x), x] \in Z(R), \quad x \in R. \]
Using similar approach as in the proof of (8) we obtain from (24) that the relation
\[ (25) \quad [f(x), y] + 2[B(x, y), x] \in Z(R), \quad x, y \in R \]
is fulfilled. Putting in (25) \( x^2 \) for \( y \) we obtain \([f(x), x^2] + 2[f(x)x + xf(x), x] \in Z(R), \quad x \in R, \) which yields \([f(x), x] + 2[f(x), x]x + 2x[f(x), x] \in Z(R), \quad x \in R. \) Hence
\[ (26) \quad 6[f(x), x]x \in Z(R), \quad x \in R. \]
From (24) and (26) we conclude that \( 6[f(x), x][x, y] = 0 \) holds for all \( x, y \in R \), which leads to
\[ (27) \quad [f(x), x][x, y] = 0, \quad x, y \in R \]
since we have assumed that \( R \) is of characteristic different from two and three.
We intend to prove that
\[ (28) \quad [f(x), x] = 0, \quad x \in R \]
is true. Obviously, we can restrict our attention on the case when \( x \notin Z(R) \).
For any fixed \( x \notin Z(R) \), a mapping \( y \mapsto [x, y] \) is a nonzero inner derivation, which means that (27) and Lemma 1 imply \([f(x), x] = 0\). Since all the requirements of Theorem 1 are fulfilled, we conclude that \( D = 0 \). The proof of the theorem is complete.

It would be interesting to know whether Theorem 2 can be proved without the assumption that \( R \) is of characteristic different from three. Theorem 1 will be used in the proof of our last result.
Theorem 3. Let $R$ be a noncommutative prime ring of characteristic different from two and three. Suppose $R$ contains the identity element $1$. Let $D: R \to R$ be an additive mapping, such that $D(x^3) = 3xD(x)x$ holds for all $x \in R$. In this case $D = 0$.

Proof. From

\begin{equation}
D(x^3) = 3xD(x)x, \quad x \in R
\end{equation}

it follows immediately

\begin{equation}
D(1) = 0.
\end{equation}

Putting in (29) $x + 1$ instead of $x$, and making use of (29) and (30), one obtains easily that $3D(x^2) = 3D(x)x + 3xD(x), x \in R$ holds. Since we have assumed that $R$ is of characteristic different from three, we have $D(x^2) = D(x)x + xD(x), x \in R$. In other words, $D$ is a Jordan derivation. We know that any Jordan derivation on a prime ring of characteristic not two is a derivation. One can replace in (29) $D(x)x$ by $D(x)x^2 + xD(x)x + x^2D(x)$, which reduces (29) to $D(x)x^2 + x^2D(x) - 2xD(x)x = 0, x \in R$. This relation can be written in the form

\[ [[D(x), x], x] = 0, \quad x \in R. \]

Therefore all the assumptions of Theorem 1 are fulfilled, which means that $D = 0$. The proof of the theorem is complete.

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References