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COMMUTING AND CENTRALIZING MAPPINGS IN PRIME RINGS

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Dedicated to the memory of my father

ABSTRACT. Let R be a ring. A mapping $F: R \to R$ is said to be commuting on R if $[F(x), x] = 0$ holds for all $x \in R$. The main purpose of this paper is to prove the following result, which generalizes a classical result of E. Posner: Let R be a prime ring of characteristic not two. Suppose there exists a nonzero derivation $D: R \to R$, such that the mapping $x \mapsto [D(x), x]$ is commuting on R. In this case R is commutative.

PRELIMINARIES

Throughout this paper R will represent an associative ring with center $Z(R)$. We write $[x, y]$ for $xy-yx$, and use the identities $[xy, z] = [x, z]y+x[y, z]$, $[x, yz] = [x, y]z + y[x, z]$. Recall that R is prime if $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping D from R to R is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. An additive mapping D from R to R is called a Jordan derivation if $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. Obviously, every derivation is a Jordan derivation. The converse is in general not true. A well-known result first proved by I. N. Herstein [5] states that every Jordan derivation on a prime ring of characteristic not two is a derivation. A brief proof of Herstein's result can be found in [2]. A derivation D is inner if there exists $a \in R$, such that $D(x) = [a, x]$ holds all for all $x \in R$. A mapping F from R to R is said to be commuting on R if $[F(x), x] = 0$ for all $x \in R$, and is said to be centralizing on R if $[F(x), x] \in Z(R)$ holds for all $x \in R$. There has been considerable interest in commuting, centralizing, and related mappings in prime and semiprime rings (see [1, 3, 4, 6, 7, 8, and 10] where further references can be found). Our methods are somewhat different from those employed by other authors.

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THE RESULTS

We shall need the following well-known and frequently used lemmas.

Lemma 1 [9, Lemma 1]. *Let $D: R \to R$ be a derivation, where R is a prime ring. Suppose that either*

$$\text{(i)}\ aD(x) = 0, \qquad x \in R$$

or

$$\text{(ii)}\ D(x)a = 0, \qquad x \in R$$

holds. In both cases we have $a = 0$ or $D = 0$.

Lemma 2 [9, Lemma 3]. *Let $D: R \to R$ be a nonzero derivation, where R is a prime ring. Suppose that D is commuting on R. In this case R is commutative.*

We shall start our investigations with our main result.

Theorem 1. *Let R be a noncommutative prime ring of characteristic not two. Suppose there exists a derivation $D: R \to R$, such that the mapping $x \mapsto [D(x), x]$ is commuting on R. In this case $D = 0$.*

A classical result in the theory of centralizing mappings is a theorem of E. Posner [9, Theorem 2] which states that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative (obviously, Lemma 2 is a special case of this result). Neglecting the fact that in the above result we have an additional assumption concerning the characteristic of the ring, we can say, that Theorem 1 generalizes Posner's theorem.

Proof of Theorem 1. We have

$$[[D(x), x], x] = 0, \qquad x \in R. \tag{1}$$

Let us introduce a mapping $B(\cdot, \cdot): R \times R \to R$ by the relation

$$B(x, y) = [D(x), y] + [D(y), x], \qquad x, y \in R.$$

It is obvious that $B(\cdot, \cdot)$ is symmetric (i.e. $B(x, y) = B(y, x)$ for all $x, y \in R$) and additive in both arguments. Moreover, a simple calculation shows that the relation

$$B(xy, z) = B(x, z)y + xB(y, z) + D(x)[y, z] + [x, z]D(y) \tag{2}$$

is fulfilled for all $x, y, z \in R$. We introduce also a mapping f from R to R by $f(x) = B(x, x)$. We have

$$f(x) = 2[D(x), x], \qquad x \in R. \tag{3}$$

Obviously, the mapping f satisfies the relation

$$f(x + y) = f(x) + f(y) + 2B(x, y), \qquad x, y \in R. \tag{4}$$

Throughout the proof we shall use the mapping $B(\cdot, \cdot)$ and the relations (2), (3), and (4) without specific reference. The relation (1) can now be written in the form

$$[f(x), x] = 0, \qquad x \in R. \tag{5}$$

The linearization of (5) gives

(6) $[f(x), y]+[f(y), x]+2[B(x, y), x]+2[B(x, y), y]=0, \qquad x, y \in R.$

The substitution $-x$ for x in the above relation leads to

(7) $[f(x), y]-[f(y), x]+2[B(x, y), x]-2[B(x, y), y]=0, \qquad x, y \in R.$

From (6) and (7) we obtain

(8) $$[f(x), y]+2[B(x, y), x]=0, \qquad x, y \in R.$$

Let us replace in (8) y by xy. Then

$$\begin{aligned} 0 &= [f(x), xy]+2[B(xy, x), x] \\ &= [f(x), xy]+2[f(x)y+xB(x, y)+D(x)[y, x], x] \\ &= [f(x), x]y+x[f(x), y]+2[f(x), x]y+2f(x)[y, x]+2x[B(x, y), x] \\ &\quad +2[D(x), x][y, x]+2D(x)[[y, x], x]=0. \end{aligned}$$

Using in the above calculation (5) and (8) we arrive at

(9) $$3f(x)[y, x]+2D(x)[[y, x], x]=0, \qquad x, y \in R.$$

Similarly, we obtain the relation

(10) $$3[y, x]f(x)+2[[y, x], x]D(x)=0, \qquad x, y \in R$$

putting in (8) yx instead of y. We intend to prove that

(11) $$3f(x)D(x)-D(x)f(x)=0, \qquad x \in R$$

holds. For this purpose we write yz instead of y in (9). We have

$$\begin{aligned} 0 &= 3f(x)[yz, x]+2D(x)[[yz, x], x] \\ &= 3f(x)[y, x]z+3f(x)y[z, x]+2D(x)[[y, x], x]z \\ &\quad +4D(x)[y, x][z, x]+2D(x)y[[z, x], x]. \end{aligned}$$

By (9) the above calculation reduces to

$$3f(x)y[z, x]+4D(x)[y, x][z, x]+2D(x)y[[z, x], x]=0, \qquad x, y, z \in R.$$

Putting in the above relation $y=D(x)$ we obtain $3f(x)D(x)[z, x]+2D(x) \times f(x)[z, x]+2D(x)^2[[z, x], x]=0$, $x, z \in R$, which yields $3f(x)D(x)[z, x] - D(x)f(x)[z, x]=0$, $x, z \in R$ according to (9). In other words we have proved the relation

(12) $$(3f(x)D(x)-D(x)f(x))[y, x]=0, \qquad x, y \in R.$$

Now we are ready for the proof of (11). There is nothing to prove if $x \in Z(R)$, since in this case $f(x)=0$. Hence we can restrict our attention on the case $x \notin Z(R)$. In this case $y \mapsto [x, y]$ is a nonzero inner derivation, which means that from (12) and Lemma 1 it follows $3f(x)D(x)-D(x)f(x)=0$. Thus the relation (11) is proved. Similarly one proves the relation

(13) $$3D(x)f(x)-f(x)D(x)=0, \qquad x \in R$$

starting from (10). From (11) and (13) one obtains easily that

$$D(x)f(x) = f(x)D(x) = 0, \qquad x \in R \tag{14}$$

holds. The linearization of the relation $D(x)f(x) = 0$ gives $0 = (D(x) + D(y)) \times (f(x) + f(y) + 2B(x, y)) = D(x)f(x) + D(y)f(x) + D(x)f(y) + D(y)f(y) + 2D(x)B(x, y) + 2D(y)B(x, y)$ which reduces to

$$D(x)f(y)+D(y)f(x)+2D(x)B(x, y)+2D(y)B(x, y) = 0, \qquad x, y \in R. \tag{15}$$

The substitution $-x$ for x in (15) gives

$$-D(x)f(y) + D(y)f(x) + 2D(x)B(x, y) - 2D(y)B(x, y) = 0, \qquad x, y \in R. \tag{16}$$

Combining (15) with (16) we arrive at

$$D(y)f(x) + 2D(x)B(x, y) = 0, \qquad x, y \in R. \tag{17}$$

Put in (17) yx for y. Then $0 = D(yx)f(x) + 2D(x)B(yx, x) = D(y)xf(x) + yD(x)f(x) + 2D(x)B(y, x)x + 2D(x)yf(x) + 2D(x)[y, x]D(x)$ which leads to

$$D(y)xf(x)+2D(x)B(x, y)x+2D(x)yf(x)+2D(x)[y, x]D(x) = 0, \quad x, y \in R$$

according to (14). The relation (17) makes us possible to write $-D(y)f(x)$ instead of $2D(x)B(x, y)$ in the above relation. Thus we have

$$D(y)[x, f(x)] + 2D(x)yf(x) + 2D(x)[y, x]D(x) = 0,$$

which yields

$$D(x)yf(x) + D(x)[y, x]D(x) = 0, \qquad x, y \in R \tag{18}$$

according to (5). Let us write in (18) xy for y. Then $0 = D(x)xyf(x) + D(x)[xy, x]D(x) = D(x)xyf(x) + D(x)x[y, x]D(x)$. Thus we have

$$D(x)xyf(x) + D(x)x[y, x]D(x) = 0, \qquad x, y \in R. \tag{19}$$

Left multiplication of the relation (18) by x gives

$$xD(x)yf(x) + xD(x)[y, x]D(x) = 0, \qquad x, y \in R. \tag{20}$$

Combining (19) with (20) we arrive at

$$f(x)yf(x) + f(x)[y, x]D(x) = 0, \qquad x, y \in R. \tag{21}$$

Our next step is to prove the relation

$$3f(x)yf(x) + 4f(x)[y, x]D(x) = 0, \qquad x, y \in R. \tag{22}$$

For this purpose we write in (10) yz instead of y. We have $0 = 3[yz, x]f(x) + 2[[yz, x], x]D(x) = 3[y, x]zf(x) + 3y[z, x]f(x) + 2[[y, x], x]zD(x) + 4[y, x] \times [z, x]D(x) + 2y[[z, x], x]D(x)$ which leads to

$$3[y, x]zf(x) + 2[[y, x], x]zD(x) + 4[y, x][z, x]D(x) = 0, \qquad x, y, z \in R \tag{23}$$

according to (10). Putting in (23) $y = 2D(x)$ and making use of (5) we arrive at $3f(x)zf(x) + 4f(x)[z, x]D(x) = 0$, $x, z \in R$ which completes the proof of (22). From (21) and (22) one obtains immediately

$$f(x)yf(x) = 0, \qquad x, y \in R$$

which implies $f(x) = 0$, $x \in R$ by primeness of R. Thus we have proved that $[D(x), x] = 0$ holds for all $x \in R$, which yields $D = 0$ by Lemma 2. The proof of the theorem is complete.

We are ready for our next result.

Theorem 2. *Let R be a noncommutative prime ring of characteristic different from two and three. Suppose there exists a derivation $D: R \to R$, such that the mapping $x \mapsto [D(x), x]$ is centralizing on R. In this case $D = 0$.*

Proof. Throughout the proof we shall use the same notation as in the proof of Theorem 1. The assumption of the theorem can be written as follows

$$[f(x), x] \in Z(R), \qquad x \in R. \tag{24}$$

Using similar approach as in the proof of (8) we obtain from (24) that the relation

$$[f(x), y] + 2[B(x, y), x] \in Z(R), \qquad x, y \in R \tag{25}$$

is fulfilled. Putting in (25) x^2 for y we obtain $[f(x), x^2] + 2[f(x)x + xf(x), x] \in Z(R)$, $x \in R$, which yields $[f(x), x]x + x[f(x), x] + 2[f(x), x]x + 2x[f(x), x] \in Z(R)$, $x \in R$. Hence

$$6[f(x), x]x \in Z(R), \qquad x \in R. \tag{26}$$

From (24) and (26) we conclude that $6[f(x), x][x, y] = 0$ holds for all $x, y \in R$, which leads to

$$[f(x), x][x, y] = 0, \qquad x, y \in R \tag{27}$$

since we have assumed that R is of characteristic different from two and three. We intend to prove that

$$[f(x), x] = 0, \qquad x \in R \tag{28}$$

is true. Obviously, we can restrict our attention on the case when $x \notin Z(R)$. For any fixed $x \notin Z(R)$, a mapping $y \mapsto [x, y]$ is a nonzero inner derivation, which means that (27) and Lemma 1 imply $[f(x), x] = 0$. Since all the requirements of Theorem 1 are fulfilled, we conclude that $D = 0$. The proof of the theorem is complete.

It would be interesting to know whether Theorem 2 can be proved without the assumption that R is of characteristic different from three. Theorem 1 will be used in the proof of our last result.

Theorem 3. *Let R be a noncommutative prime ring of characteristic different from two and three. Suppose R contains the identity element 1. Let $D: R \to R$ be an additive mapping, such that $D(x^3) = 3xD(x)x$ holds for all $x \in R$. In this case $D = 0$.*

Proof. From

$$D(x^3) = 3xD(x)x, \qquad x \in R \tag{29}$$

it follows immediately

$$D(1) = 0. \tag{30}$$

Putting in (29) $x + 1$ instead of x, and making use of (29) and (30), one obtains easily that $3D(x^2) = 3D(x)x + 3xD(x)$, $x \in R$ holds. Since we have assumed that R is of characteristic different from three, we have $D(x^2) = D(x)x + xD(x)$, $x \in R$. In other words, D is a Jordan derivation. We know that any Jordan derivation on a prime ring of characteristic not two is a derivation. One can replace in (29) $D(x^3)$ by $D(x)x^2 + xD(x)x + x^2D(x)$, which reduces (29) to $D(x)x^2 + x^2D(x) - 2xD(x)x = 0$, $x \in R$. This relation can be written in the form

$$[[D(x), x], x] = 0, \qquad x \in R.$$

Therefore all the assumptions of Theorem 1 are fulfilled, which means that $D = 0$. The proof of the theorem is complete.

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