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LARGE COMPACT SEPARABLE SPACES MAY ALL CONTAIN βN

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ABSTRACT. In the Cohen model any compact separable space that does not contain βN has cardinality at most of the continuum.

In this paper we prove the theorem stated in the abstract. By the Cohen model we mean any model obtained by adjoining $\aleph_2$ Cohen reals to a model of GCH. We shall, in fact, show that any compact separable space of cardinality greater than the continuum c will map onto the space I^c. We will see that this is equivalent to containing βN. We freely use any notation and conventions which we believe to be standard enough to justify this usage.

Suppose that V is a model of GCH, G is $Fn(\omega_2, 2)$-generic over V, and that in $V[G]$, X is a compact separable space with cardinality greater than c. We will assume (and show later that we may assume) that X has a countable discrete dense set ω.

In V, let θ be a large enough regular cardinal and let M be an elementary submodel of $H(\theta)$ such that $X \in M[G]$, $|M| = \aleph_2$ and such that $M^{\omega_1} \subset M$. It follows that $M[G]^{\omega_1} \subset M[G]$. Still in V, choose an elementary chain $\{M_\alpha | \alpha < \omega_2\}$ of elementary submodels of M whose union is M and so that each submodel is closed under ω-sequences and each has cardinality $\aleph_1$. It is remarked in [2] that since $M \prec H = H(\theta)$, it will be the case that $M[G] \prec H[G]$ in $V[G]$. This is easily proven using names, forcing, and the definition

$$M[G] = \{\text{val}\,(\dot\tau, G) | \dot\tau \in M \text{ is a } Fn(\omega_2, 2)\text{-name}\}.$$

It will also be the case here (although not in general) that $H[G]$ will be $H(\theta)$ as computed in $V[G]$.

Passing to $V[G]$, choose a point $x \in X \backslash M$. For each $\alpha < \omega_2$, define

$$x^\alpha = \{Y \in \mathscr{P}(\omega) \cap M_\alpha[G] | x \in \text{int}_X \text{cl}_X Y\}.$$

Although x is not in $M[G]$, it is in $H[G]$, and the set x^α is just an ω_1-sized subset of $M_\alpha[G]$ and is therefore in $M[G]$. By elementarity, it follows that

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there is some point $x_\alpha \in M[G]$ such that

$$x^\alpha = \{Y \in \mathscr{P}(\omega) \cap M_\alpha[G] | x_\alpha \in \operatorname{int}_X \operatorname{cl}_X Y\}.$$

Now choose a continuous function $f_\alpha \colon X \to I$ so that

$$x_\alpha \in \operatorname{int}_X \operatorname{cl}_X[f_\alpha^{-1}(0) \cap \omega] \text{ and } x \in \operatorname{int}_X \operatorname{cl}_X[f_\alpha^{-1}(1) \cap \omega].$$

Let $A_\alpha = f_\alpha^{-1}(0) \cap \omega$ and $B_\alpha = f_\alpha^{-1}(1) \cap \omega$. Since M contains (up to forcing isomorphism) all $Fn(\omega_2, 2)$-names of subsets of ω, it follows that $\{A_\alpha, B_\alpha\} \subset M[G]$.

Now, returning to V, we may fix names for all of the elements of $V[G]$ already mentioned. By thinning out and reindexing, we may assume that:

1. $\dot{A}_\alpha$ and $\dot{B}_\alpha$ are $Fn(I_\alpha, 2)$-names for some $I_\alpha \in [\omega_2]^\omega$;
2. $\{I_\alpha | \alpha \in \omega_2\}$ forms a Δ-system with root I;
3. for each $\alpha < \beta < \omega_2$ there is an isomorphism $\phi_{\alpha,\beta}$ taking $\dot{A}_\alpha$, $\dot{B}_\alpha$ to $\dot{A}_\beta$ and $\dot{B}_\beta$, respectively, such that $\phi_{\alpha,\beta}$ is induced by a bijection from I_α to I_β which is the identity on I;
4. each of $\dot{x}_\alpha$, $\dot{A}_\alpha$, and $\dot{B}_\alpha$ are in $M_{\alpha+1}$; and
5. x^α is equal $\{Y \in M_\alpha[G] | x_\alpha \in \operatorname{cl}_X \operatorname{int}_X Y\}$.

We complete the proof by showing that

Claim. $V[G] \vDash \{(A_\alpha, B_\alpha) | \alpha < \omega_2\}$ is a dyadic family. (I.e., for any disjoint finite $F, H \subset \omega_2$, $\bigcap_{\alpha \in F} A_\alpha \cap \bigcap_{\alpha \in H} B_\alpha \neq \varnothing$.)

The main idea here is that since x_β is chosen to be like x, we can often force that $x_\beta \in \operatorname{cl}_X B_\alpha$ for $(\beta > \alpha)$ and by thinning out, we have that x_β is like x_α, so we can often force that $x_\beta \in \operatorname{cl}_X A_\alpha$.

Definition 1. Let $G_1 = G \cap M_1$. For any $Fn(\omega_2, 2)$-name $\dot{Y}$ and condition $r \in Fn(\omega_2, 2)$, let

$$\dot{Y}_r = \{n \in \omega | (\exists q < r)(q|I \in G_1 \text{ and } q \Vdash n \in \dot{Y})\}.$$

Although it is an abuse of notation, we will think of $\dot{Y}_r$ as being an element of $M[G_1]$, as well as being a $Fn(\omega_2 \cap M_1, 2)$-name of itself.

Proposition 1. *For any $r \in Fn(\omega_2, 2)$ such that $r \cap M_1 \in G_1$ and any $\alpha \in \omega_2$,*

$$V[G] \vdash \dot{A}_{\alpha,r} \in x^1.$$

Proof. It should be clear from Definition 1 that $\dot{A}_{\alpha,r} = \dot{A}_{\alpha,(r|I_\alpha)}$ (since we may assume $q \in Fn(I_\alpha, 2)$). Now let $r' \in Fn(I_0, 2)$ be such that $\phi_{0,\alpha}(r') = r|I_\alpha$ (if $\alpha = 0$, let $\phi_{0,\alpha}$ be the identity function). Now observe that, since $\dot{A}_\alpha$ is isomorphic to $\dot{A}_0$ by $\phi_{0,\alpha}$,

$$\dot{A}_{\alpha,r} = \dot{A}_{0,r'} \in M_1[G_1].$$

Note that $\{\phi_{0,\beta}(r') | \beta \in \omega_2\}$ is what one might call a Δ-system of conditions with root $r'|I_0$, i.e., they each extend $r'|I_0$ and their domains form a Δ-system with root equal to the domain of $r'|I_0$. Such a Δ-system is always "dense-below" the root. Since $r'|I_0 \in G$, we may choose β such that $r_\beta = \phi_{0,\beta}(r') \in G$.

Again, note that $\dot{A}_{\beta, r_\beta} = \dot{A}_{0, r'}$. Now it is clear that (in $V[G]$)

$$x_\beta \in \mathrm{cl}_X \, \mathrm{int}_X[\dot{A}_{\beta, r_\beta}]$$

since

$$r_\beta \in G \text{ and } r_\beta \Vdash \dot{A}_\beta \subset \dot{A}_{\beta, r_\beta}.$$

Therefore, we have $x_\beta \in \mathrm{cl}_X \, \mathrm{int}_X[\dot{A}_{0, r'}]$, from which it follows that $\dot{A}_{\alpha, r} = \dot{A}_{0, r'} \in x^\beta$. The result now follows from the fact that $x^\beta \cap M_1[G_1] = x^1$.

Proposition 2. *For any* $r \in Fn(\omega_2, 2)$ *such that* $r|I \in G_1$ *and any* $\alpha < \omega_2$,

$$V[G] \vDash \dot{B}_{\alpha, r} \in x^1.$$

Proof. As shown earlier, we know that $\dot{B}_{\alpha, r} \in M_1[G_1]$ and we may find a $\beta > \alpha$ so that $r_\beta = \phi_{\alpha, \beta}(r|I_\alpha) \in G$. Now, by the isomorphism, it follows that $\dot{B}_{\alpha, r} = \dot{B}_{\beta, r_\beta}$. Since $\alpha < \beta$, we also have that $\dot{B}_{\alpha, r} \in M_\beta[G_\beta]$. Again,

$$r_\beta \in G \text{ and } r_\beta \Vdash \dot{x} \in \mathrm{cl}_X(\dot{B}_\beta) \subset \mathrm{cl}_X \, \mathrm{int}_X[\dot{B}_{\beta, r_\beta}].$$

Proposition 3. $V[G] \vDash x^1$ *contains all the cofinite sets and has the finite intersection property.*

Proposition 4. $V[G] \vDash$ *for any* $r \in Fn(\omega_2, 2)$ *such that* $r|I \in G$ *and any disjoint* $F, H \in [\omega_2]^{<\omega}$.

$$\bigcap_{\alpha \in F} \dot{A}_{\alpha, r} \cap \bigcap_{\alpha \in H} \dot{B}_{\alpha, r} \neq \varnothing.$$

Proof. By Propositions 1 and 2, each of these sets are in x^1. Now apply Proposition 3.

Proof of claim. Suppose that $r \in G$ and $F, H \in [\omega_2]^{<\omega}$ are such that

$$F \cap H = \varnothing \text{ and } r \Vdash \bigcap_{\alpha \in F} \dot{A}_\alpha \cap \bigcap_{\alpha \in H} \dot{B}_\alpha = \varnothing.$$

By Proposition 4, we may choose an integer k so that

$$k \in \bigcap_{\alpha \in F} \dot{A}_{\alpha, r} \cap \bigcap_{\alpha \in H} \dot{B}_{\alpha, r}.$$

For each $\alpha \in F$, we may choose $r_\alpha \in Fn(I_\alpha, 2)$ so that

$$r_\alpha < r|I_\alpha, \; r_\alpha|I \in G \text{ and } r_\alpha \Vdash k \in \dot{A}_\alpha.$$

Similarly for each $\alpha \in H$, we may choose $r_\alpha \in Fn(I_\alpha, 2)$ so that

$$r_\alpha < r|I_\alpha, \; r_\alpha|I \in G \text{ and } r_\alpha \Vdash k \in \dot{B}_\alpha.$$

Since $\{I_\alpha | \alpha \in \omega_2\}$ is a Δ-system with root I, it follows that $q = r \cup \bigcup_{\alpha \in F \cup H} r_\alpha$ is in $Fn(\omega_2, 2)$. Furthermore,

$$q \Vdash k \in \bigcap_{\alpha \in F} \dot{A}_\alpha \cap \bigcap_{\alpha \in H} \dot{B}_\alpha,$$

which is a contradiction.

Therefore we have established the following theorem.

Theorem. *In* $V[G]$, *if* X *is a compact separable space, then*

$$|X| > c \Leftrightarrow X \text{ maps onto } I^c.$$

(*If we assume that* X *is* 0-*dimensional, then we can replace* I^c *by* 2^c *in the statement.*)

Proof. Since X is separable, it can be embedded in a compactification $b\omega$ of ω in such a way that $[b\omega - \omega] = X$. Let $\{f_\alpha | \alpha \in \omega_2\}$ be the family of maps chosen as above, i.e., $\{(f_\alpha^{-1}(0) \cap \omega, f_\alpha^{-1}(1) \cap \omega) | \alpha \in \omega_2\}$ form a dyadic family. Therefore, the function $\prod_{\alpha \in \omega_2} f_\alpha$ is a continuous function from $b\omega$ into I^c. Since the family $\{(f_\alpha^{-1}(0) \cap \omega, f_\alpha^{-1}(1) \cap \omega) | \alpha \in \omega_2\}$ is dyadic, it follows that the image of $b\omega$ contains 2^c. Also, there is a continuous map from I^c to itself which takes the subspace 2^c onto I^c. Now it follows easily that the restriction of the composition of the two maps to X is an onto map.

Our primary motivation for proving the above result was to show that there need not be a "psi-space" with a compactification of cardinality greater than c. This follows from the above result by a result of Baumgartner and Weese that is equivalent to the result that, in the above model, no (zero-dimensional) compactification of a ψ-space can be mapped onto 2^c. A ψ-space is a space containing a countable dense set of isolated points such that the set of nonisolated points is a discrete set and every infinite set of isolated points contains a converging subsequence. In [1] a Boolean algebra, $\mathscr{B}$, is called representable if there is a maximal almost-disjoint family $\mathscr{A}$ of subsets of ω so that $\mathscr{B}$ is isomorphic to the quotient of $\mathscr{P}(\omega)$ by the ideal generated by $\mathscr{A}$. This corresponds to the fact that the Stone space of $\mathscr{B}$ is a compactification of a ψ-space. The following lemma is the last part of their Theorem 4.1 (modified so as to include spaces which are not zero-dimensional).

Lemma [1]. *In* $V[G]$, *if* ω *is densely embedded into* I^c *by a function* f, *then there is a set* $Y \subset \omega$ *such that* $f[Y]$ *contains no converging sequences. Hence, no* ψ-*space maps continuously to a dense subset of* I^c.

Proof. Suppose that $\dot{f}$ is the name of f. For each $\alpha \in \omega_2$, let $\dot{A}_\alpha$ be the name of $\{n \in \omega | f(n)(\alpha) < \frac{1}{4}\}$. Similarly let $\dot{B}_\alpha$ be the name for $\{n \in \omega | f(n)(\alpha) > \frac{3}{4}\}$. Also for each α, fix a countable set $I_\alpha \subset \omega_2$, so that both $\dot{A}_\alpha$ and $\dot{B}_\alpha$ are $Fn(I_\alpha, 2)$-names. Note that we may assume that

$$I \Vdash \{(\dot{A}_\alpha, \dot{B}_\alpha) | \alpha \in \omega_2\} \text{ is a dyadic family.}$$

Let M be an $\aleph_1$-sized elementary submodel of a sufficiently large $H(\theta)$ such that $\{\dot{f}\} \cup \{(\dot{A}_\alpha, \dot{B}_\alpha, I_\alpha) | \alpha \in \omega_2\} \in M$ and M is closed under ω-sequences. Let $M \cap \omega_2 = \lambda$ and let $I = I_\lambda \cap M$. Since M is an elementary submodel of $H(\theta)$, we can inductively choose $\{\alpha_n | n \in \omega\} \subset \lambda$ so that:

1. $\dot{A}_{\alpha_n}$ and $\dot{B}_{\alpha_n}$ are isomorphic to $\dot{A}_\lambda$ and $\dot{B}_\lambda$ by an isomorphism ϕ_n induced by the order preserving isomorphism from I_{α_n} to I_λ, which is the identity on I.
2. $I_{\alpha_{n+1}} \cap \sup(I_{\alpha_n}) = I$ for each $n \in \omega$.

We can choose in $M[G \cap M]$ an infinite $Y \subset \omega$ and a name $\dot{Y} \in M$ for Y so that Y is almost contained in $A_{\alpha_{2n}} \cap B_{\alpha_{2n+1}}$ for each $n \in \omega$ (recall that the family is dyadic). If Y contains a subset whose image under f converged, there would be such a set in $M[G]$. Therefore we may assume that $f[Y]$ converges and we finish the proof by showing that

$$A_\lambda \cap Y \text{ and } B_\lambda \cap Y \text{ are both infinite.}$$

Indeed, suppose $p \in G$ and $j \in \omega$ are such that

$$p \Vdash \dot{A}_\lambda \cap \dot{Y} \subset j.$$

Choose n large enough so that $[I_{\alpha_{2n}} - I] \cap \operatorname{dom}(p) = \emptyset$. Let $r \in Fn(I_{\alpha_{2n}}, 2)$ exist such that $\phi_{2n}(r) = p|I_\lambda$; note that r is compatible with p. Now

$$(r \cup p) \Vdash \dot{Y} \cap A_{\alpha_{2n}} \text{ is infinite},$$

so we may choose $k > j$ and $r' < r \cup (p \cap M)$ with $r' \in M$ so that $r' \Vdash k \in \dot{Y} \cap A_{\alpha_{2n}}$. But now $\phi_{2n}(r'|I_{2n}) = p' \Vdash k \in A_\lambda$ and p' is compatible with r'. To see this last fact we note that $r' < r$, $\phi_{2n}(r|I_{2n}) < p$, and $r' < p' \cap M = p' \cap \operatorname{dom}(r')$.

The case $p \Vdash \dot{B}_\lambda \cap \dot{Y} \subset j$ is obviously similar.

References

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