NOTES ON THE BAUMSLAG–SOLITAR NONRESIDUALLY FINITE EXAMPLES

ROBERT I. CAMPBELL

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Abstract. We examine the abelianization of G. Baumslag and Solitar’s example of a one-generator group that is not residually finite. In particular, the nonfinitely-generated commutator subgroup is shown to be not residually finite. We also review a specific example of a cyclic extension of a residually finite group that is not residually finite.

Theorem 1. If the sequence \( 1 \rightarrow N \rightarrow E \rightarrow \mathbb{Z} \rightarrow 1 \) is exact, where \( N \) is finitely generated and residually finite, then \( E \) is residually finite.

A proof of this result may be found in Hempel [Hem, Corollary 15.21, p. 180]. More general forms of this result, replacing \( \mathbb{Z} \) with any residually finite group and requiring that the sequence split, were proved by Mal’cev [Mal] and Miller [Mil, Theorem III.7, p. 29].

I originally conjectured that this result is still true even if we drop the condition that \( N \) is finitely generated. The first example in this paper is part of a proposed counterexample to this conjecture, and the second example is a simple counterexample which was pointed out to me by Geoff Mess. This second example also follows from work done by Gruenberg [Gruen].

Consider the example given by G. Baumslag and Solitar [BS] of a one-relator group that is not residually finite: \( \langle a, b \mid a^{-1}b^2a = b^3 \rangle \). Abelianizing this group maps \( b \mapsto 1 \) and yields \( \mathbb{Z} \), the free group with a single generator \( a \). We will refer to the kernel of this abelianization as \( N \). If we define \( b_i \equiv a'ba^{-1} \) then \( N \) has the following explicit (neither finitely presented nor finitely generated) presentation:

\[ N = \langle \ldots b_{-1}, b_0, b_1, b_2, \ldots \mid b_i^2 = b_{i+1}^3 \rangle. \]

This group fits into the exact sequence

\[ 1 \rightarrow N \rightarrow BS \text{ Group} \rightarrow \mathbb{Z} \rightarrow 1. \]
If $N$ is residually finite, this will be an example of a semidirect product of a residually finite group by $\mathbb{Z}$ which is not itself residually finite. We prove that this is not the case:

**Theorem 2.** $N$ is not residually finite.

In fact, we show that the only finite quotients of $N$ are cyclic. Thus, if $g \in [N, N]$, then for any finite representation, $N \rightarrow \Gamma$, we get $\alpha(g) = 1$. We will need a technical lemma, whose proof we defer until after the proof of the theorem.

**Lemma.** Let $N \rightarrow \Gamma$, where $\Gamma$ is a finite group, and let $\gamma_i \equiv \alpha(b_i)$. Then for all $i$, we get that $3$ does not divide the order of $\gamma_i$ and $2$ does not divide the order of $\gamma_i$.

**Proof of Theorem 2.** Assume that there is a map $N \rightarrow \Gamma$ which maps $N$ to a finite group. From Lemma 1 we learn that for all $i$ the order of $\gamma_i$ is some $m$ where $3$ does not divide $m$ and $2$ does not divide $m$. As $3 \nmid m$ we see that $\gamma_i^3$ is a generator of $\langle \gamma_i \rangle \cong \mathbb{Z}_m$. Similarly, $\gamma_i^2$ is a generator of $\langle \gamma_i+1 \rangle \cong \mathbb{Z}_m$. Thus, $\gamma_i^3 = \gamma_{i+1}^2$ and we see that $\langle \gamma_i \rangle = \langle \gamma_{i+1} \rangle$. By continuing this process, we see that $N = \langle \gamma_0 \rangle \cong \mathbb{Z}_m$, the cyclic group of order $m$. As this group is abelian, the kernel of the projection of $N$ onto this quotient includes the commutator subgroup of $N$. Thus, all finite index normal subgroups of $N$ include the commutator subgroup, and, in particular, their intersection is not empty. Hence $N$ is not residually finite.

We now prove the lemma:

**Proof of lemma.** Recall that $\gamma_i \equiv \alpha(b_i)$. Let $o(\Gamma) = 3^k 2^L M$, where $3 \nmid M$ and $2 \nmid M$. As $o(\gamma_i) \mid o(\Gamma)$, then if $o(\gamma_i) = 3^k 2^l m_j$ where $3 \nmid m_j$ and $2 \nmid m_j$, we find that $k_j \leq K$ and $l_j \leq L$.

**Claim.** $\forall i \exists \exists o(\gamma_i)$

Assume the opposite, so for some $i$, $3^{k_i} \mid o(\gamma_i)$ where $k_i > 0$. We now show that for any $j \leq i$, we have $k_{j-1} = k_j + 1$ and hence, by induction, $k_j = k_i + (i - j)$.

**Case I.** $2 \nmid o(\gamma_j)$.

$\Rightarrow o(\gamma_{j-1}) = 2 \cdot o(\gamma_j)/2$.

This holds as $1 = \gamma_j^{3^i 2^l m_j} = (\gamma_j^3)^{3^i 2^l - 1} m_j = (\gamma_{j-1}^{3^i 2^l - 1} m_j = \gamma_{j-1}^{3^i 2^l - 1} m_j$.

**Case II.** $2 \mid o(\gamma_j)$, so $o(\gamma_j^3) = o(\gamma_j)$. However, since $\gamma_{j-1}^3 = \gamma_j^2$, we have $o(\gamma_{j-1}^3) = o(\gamma_j^2) = 3^{k_j} m_j \Rightarrow o(\gamma_{j-1}) = 3 \cdot o(\gamma_{j-1}) = 3^{k_{j+1}} m_j$. Again, this holds for all $k_j > 0$.

But $k_j \leq K$ is a finite bound for $k_j$. This contradiction yields that $\forall i \exists \exists o(\gamma_i)$, proving the above claim. Similarly, one may show that $\forall i 2 \nmid o(\gamma_i)$.
Now, if both $2 \nmid o(\gamma_i)$ and $3 \nmid o(\gamma_i)$ we have $o(\gamma_{i-1}) = o(\gamma_i^3) = o(\gamma_i^2) = o(\gamma_i) = m$, so we may drop the subscript $i$ to get $\forall j \; o(\gamma_j) = m$, where $3 \nmid m$, $2 \nmid m$.

More generally, Baumslag and Solitar [BS] produced an entire class of one-relator groups which are not residually finite. The nonzero integers $p$ and $q$ are said to be meshed if either $p$ or $q$ divides the other or if $p$ and $q$ have precisely the same set of prime divisors.

**Theorem 3 (Baumslag–Solitar).** Let $p$ and $q$ be nonzero integers. Then

$$G_{p,q} \equiv \langle a, b \mid a^{-1}b^pa = b^q \rangle$$

is Hopfian if and only if $p$ and $q$ are meshed.

Note that a result of Mal’cev [MKS, p. 415] is that any finitely generated residually finite group is Hopfian. Thus, for $p$ and $q$ not meshed, $G_{p,q}$ is not Hopfian and, as it is finitely generated, $G_{p,q}$ is not residually finite. We further note that if we denote the commutator of $G_{p,q}$ by $N_{p,q}$ these groups fall into the exact sequence

$$1 \rightarrow N_{p,q} \rightarrow G_{p,q} \rightarrow \mathbb{Z} \oplus \mathbb{Z}_{|p-q|} \rightarrow 1.$$  

$N_{p,q}$ has the presentation

$$N = \langle \ldots, b_{-1}, b_0, b_1, b_2, \ldots | b_i = b_{i+1}^{p} \rangle$$

where $b_i \equiv a^i b^{p-q} a^{-i}$. The following theorem may be proven by a simple rewrite of the proof of Theorem 2:

**Theorem 4.** If $p$ and $q$ are mutually prime, then $N_{p,q}$ is not residually finite.

We can now handle the more general case of $p$ and $q$ not meshed by reducing it to the case of $p$ and $q$ mutually prime, as shown previously. The method used to do this was suggested by G. Baumslag.

**Theorem 5.** If $p$ and $q$ are not meshed then the group $N_{p,q}$ is not residually finite.

**Proof.** If $p$ and $q$ are mutually prime, then we have the case dealt with in Theorem 4, so we will assume that $\gcd(p,q) = r \neq 1$. Define $P$ as $p/r$ and $Q$ as $q/r$. Consider the subgroup of $N_{p,q}$ generated by $\{ b_i^r \}$, which is isomorphic to $N_{P,Q}$. As $\gcd(P, Q) = 1$ we have reduced the problem to that dealt with in Theorem 4 above, so $N_{p,q}$ is not residually finite. Any subgroup of a residually finite group is itself residually finite, so $N_{p,q}$ is not residually finite.

The following example of a non-residually finite cyclic extension of a residually finite group was suggested by Geoff Mess: Consider the wreath product of the alternating group $A_5$ by $\mathbb{Z}$. This may be considered as an extension:

$$1 \rightarrow \bigoplus_{i \in \mathbb{Z}} A_5 \rightarrow A_5 \rightarrow \mathbb{Z} \rightarrow 1.$$
We note that while \( \bigoplus_{i \in \mathbb{Z}} A_5 \) is obviously residually finite, we can prove that \( A_5 \wr \mathbb{Z} \) is not.

**Theorem 6.** \( A_5 \wr \mathbb{Z} \) is not residually finite.

**Proof.** We use the notation \( (A_5)^{i+1} = i(A_5)^{i}i^{-1} \) to describe the action of \( \mathbb{Z} \) on our wreath product. Assume that there is some homomorphism \( \alpha : A_5 \wr \mathbb{Z} \to \Gamma \), where \( \Gamma \) is finite and nontrivial. If we look at the image of the generator of \( \mathbb{Z} \) we see that there is some least integer \( n \), such that \( \alpha(i^n) = 1 \). As \( A_5 \) is simple, \( \alpha((A_5)_i) \) must be either trivial or isomorphic to \( A_5 \). Now find an integer \( i \) such that the image of \( A_5 \) is nontrivial. (If none exists then the image of \( \alpha \) is cyclic, hence abelian, and we are done.) We get \( \alpha((A_5)^{i+n}) = \alpha((A_5)_i) \). We note that from our construction of the wreath product that \( (A_5)^{i+n} \) must be in the centralizer of \( (A_5)_i \), but the center of \( A_5 \) is trivial. Thus \( \alpha((A_5)^{i+n}) \) must be trivial, contradicting our assumption that we could find \( i \) such that \( \alpha((A_5)_i) \) is not trivial. Thus \( \Gamma \) must be cyclic, and the wreath product is not residually finite (in fact, having only cyclic quotients).

This result is also a consequence of work by Gruenberg [Gruen, Theorem 3.1].

**Theorem 7.** Let \( \mathcal{P} \) be any property satisfying the condition that whenever a group has \( \mathcal{P} \), then all its subgroups also have \( \mathcal{P} \). If \( W = G \wr \Gamma \) is residually \( \mathcal{P} \) where \( \Gamma \) is transitive, then either \( \Gamma \) is \( \mathcal{P} \) or \( G \) is abelian.

**References**


Mathematics Department, University of California, Berkeley, California 94720

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