CONVOLUTION IN THE HARMONIC HARDY CLASS $h^p$ WITH $0 < p < 1$

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Abstract. It is proved that if $u \in h^p$, $0 < p < 1$, and $v \in h^q$, $q > p$, then
$$M_q(u * v, r) = 0 \left( (1 - r)^{1 - 1/p} \right), \quad r \to 1^-,$$
where $u * v$ stands for the convolution of $u$ and $v$.

1. Introduction

Let $h(\Delta)$ denote the class of all (complex-valued) harmonic functions in the unit disc $\Delta$ of the complex plane $\mathbb{C}$. The harmonic Hardy class $h^p$, $0 < p < \infty$, is the subclass of $h(\Delta)$ consisting of those functions $u$ for which
$$\|u\|_p := \sup_{0 < r < 1} M_p(u, r) < \infty.$$ Here $M_p(u, r)$ is the $p$th mean value of $u$:
$$M^p_p(u, r) = \int_T |u(rw)|^p d\delta(w),$$
where $T$ is the unit circle and $d\delta$ is the normalized Lebesgue measure on $T$.

The Hardy space $H^p$ is the closed subspace of $h^p$ consisting of functions holomorphic in $\Delta$.

For $p \geq 1$, the structure of $h^p$ spaces is well known (see [1], Chapters 2, 3, and 4) and is similar to the structure of $H^p$ and $L^p$ spaces. The fundamental results concerning the case $p < 1$ were obtained by Hardy and Littlewood [2]; for further information and references we refer to [5].

Each $u \in h(\Delta)$ has a unique series expansion
$$u(rw) = \sum_{k = -\infty}^{\infty} \hat{u}(k)r^{|k|}w^k, \quad w \in T, \quad 0 < r < 1,$$
which converges uniformly and absolutely on compact subsets of $\Delta$. The convolution of $u$ and $v(\in h(\Delta))$ is defined by
$$(u * v)(rw) = \sum_{k = -\infty}^{\infty} \hat{u}(k)\hat{v}(k)r^{|k|}w^k, \quad w \in T, \quad 0 < r < 1,$$
which can be written as
\[(1.1) \quad (u \ast v)(r^2 w) = \int_T u(rw \xi)v(r \xi) \, d\delta(\xi), \quad w \in T, \quad 0 < r < 1.\]

This identity along with Minkowski’s inequality in continuous form is used to prove that
\[M_p(u \ast v, r^2) \leq M_p(u, r)M_1(v, r), \quad p \geq 1.\]

In particular, if \(u \in h^p, \ p \geq 1,\) and \(v \in h^1,\) then \(u \ast v \in h^p.\) This fact is extended by the following theorem.

**Theorem 1.1.** If \(0 < p < 1\) and \(q \geq p,\) then there is a constant \(C = C_{p, q} < \infty\) such that
\[(1.2) \quad M_q(u \ast v, r) \leq C (1 - r)^{1 - 1/p} \|u\|_p \|v\|_q, \quad 0 < r < 1,\]
for all \(u \in h^p, \ v \in h^q.\)

The analytic case of this theorem is considered in [3], and the proof is based on the inequality
\[(1.3) \quad M(F, r) \leq C_p (1 - r)^{1 - 1/p} M_p \left( F, \frac{1 + r}{2} \right), \quad (p < 1),\]
which is valid for analytic functions.

In §1 we use a certain maximal function to obtain a more general version of (1.3). However, the heart of our proof of Theorem 1.1 is the following fundamental result of Hardy and Littlewood [2].

**Theorem HL.** Let \(U\) be a harmonic function in a disc \(D\) centered at \(z_0.\) Then, for each \(p > 0,\)
\[|U(z_0)|^p \leq C_p m(D)^{-1} \int_D |U|^p \, dm,\]
where \(C_p\) is a constant depending only on \(p,\) and \(dm\) stands for the Lebesgue measure in the plane.

A short proof of Theorem HL is given in [4].

Theorem 1.1 is a consequence of a stronger result that will be proved in §3. Also, §3 contains a generalization of Theorem 1.1 to a class of weighted \(h^p\) spaces.

**2. A MAXIMAL FUNCTION**

For a complex-valued function \(F\) continuous in the unit disc \(\Delta\) we define the function \(F^+\) by
\[(2.1) \quad F^+(r \xi) = \sup \{|F(rz)| : |z - \xi| \leq 1 - r, \ z \in T\}, \quad \xi \in T, \ r < 1.\]
(Note that the supremum is taken over an arc.) This maximal function differs from (and is much simpler than) the radial maximal function of Hardy and Littlewood.
Lemma 2.1. Let $0 < p < 1$, and let $F$ be a function continuous in $\Delta$. Then

\[(2.2) \quad M_1(F, r) \leq C(1 - r)^{1 - 1/p} M_p(F^+, r), \quad 0 < r < 1,
\]

where $C$ depends only on $p$.

Proof. First we prove that

\[(2.3) \quad M_\infty (F, r) \leq C (1 - r)^{-1/p} M_p(F^+, r), \quad 0 < r < 1,
\]

where

\[M_\infty (F, r) = \sup \{|F(rw)| : w \in T\} .\]

For a fixed $r$, $0 < r < 1$, let $T_\zeta = \{w \in T : |w - \zeta| \leq 1 - r\}$. It follows from (2.1) that if $w \in T_\zeta$, then

\[F^+(rw) = \sup \{|F(rz)| : w \in T_\zeta\} \geq |F(r\zeta)| .\]

Hence

\[M^p p(F^+, r) = \int_{T_\zeta} F^+(rw)^p d\delta(w) \geq \int_{T_\zeta} |F(r\zeta)|^p d\delta(w) = |F(r\zeta)|^p \delta(T_\zeta) .\]

But $\delta(T_\zeta)$ is independent of $\zeta$ and equals $\frac{2}{\pi} \arcsin(\frac{1 - r}{2})$, whence

\[M^p p(F^+, r) \geq |F(r\zeta)|^p (1 - r)/\pi ,\]

which proves (2.3). We see that (2.3) holds for all $p > 0$. If $p < 1$, then

\[M_1(F, r) \leq M_\infty^{1-p}(F, r) M_p(F, r) \leq M_\infty^{1-p}(F, r) M_p(F^+, r) ,\]

and now (2.2) follows from (2.3). \qed

When $p \geq 1$, there is a simple relation between the $p$th mean values of a harmonic function $U$ and $U^+$. Namely, it is easy to see that for the Poisson kernel $P$, we have $P^+ \leq CP$, which implies that

\[(2.4) \quad M_p \left(U^+, r\right) \leq CM_p \left(U, \frac{1 + r}{2}\right) \quad (p \geq 1)
\]

where $C$ is an absolute constant. In the case $p < 1$ we must use some other mean values. For $0 < p < \infty$ we let

\[A^p_p(F, r) = \left\{\left(1 - r\right)^{-2} \int_{r_0}^{r_1} M^p_p(F, \varphi) d\varphi\right\}^{1/p} ,\]

where

\[r_0 = \max\{0, r - (1 - r)/2\}, \quad r_1 = r + (1 - r)/2 = (1 + r)/2, \quad 0 < r < 1 .\]

Note that $A^p_p(F, r)$ is "equivalent" to the $p$th mean value of $F$ over the annulus $r_0 \leq |z| \leq r_1$.  

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Lemma 2.2. If $p > 0$ and if $U$ is harmonic in $\Delta$, then there is a constant $C = C_p$ such that

\begin{equation}
M_p \left( U^+, r \right) \leq CA_p(U, r), \quad 0 < r < 1.
\end{equation}

This implies (2.4) because of the increasing property of $M_p(U, r)$, $p \geq 1$.

Proof. For fixed $r$ and $z \in T$ let $D = \{ w \in \mathbb{C} : |w - rz| \leq (1 - r)/2 \}$. By Theorem HL, we have

\begin{equation}
|U(rz)|^p \leq C (1 - r)^{-2} \int_D |U(w)|^p \, dm(w).
\end{equation}

If $w \in D$, then

\begin{equation}
|1 - rw\overline{z}| \leq (3/2)(1 - r).
\end{equation}

From this and (2.6) we find that

\begin{equation}
|U(rz)|^p \leq C \int_D |U(w)|^p \, |1 - rw\overline{z}|^{-2} \, dm(w).
\end{equation}

Since $D$ is contained in the annulus

\begin{equation}
A(r) = \{ w \in \mathbb{C} : r - (1 - r)/2 \leq |w| \leq r + (1 - r)/2 \}
\end{equation}

we get

\begin{equation}
|U(rz)|^p \leq C \int_{A(r)} |U(w)|^p \, |1 - rw\overline{z}|^{-2} \, dm(w), \quad z \in T,
\end{equation}

where $C$ depends only on $p$. On the other hand, if $|z| = |\zeta| = 1$ and $|z - \zeta| \leq 1 - r$, then for all $w \in \Delta$, we have

\begin{equation}
|1 - rw\overline{\zeta}| \leq 2 |1 - rw\overline{z}|.
\end{equation}

From this, (2.7), and (2.1) we see that

\begin{equation}
U^+ (r\zeta)^p \leq C \int_{A(r)} |U(w)|^p \, |1 - rw\overline{\zeta}|^{-2} \, dm(w), \quad |\zeta| = 1.
\end{equation}

Hence, by integration over the circle $|\zeta| = 1$,

\begin{equation}
M_p^\ast \left( U^+, r \right) \leq C \int_{A(r)} |U(w)|^p \, (1 - r|w|)^{-1} \, dm(w).
\end{equation}

Integration in polar coordinates concludes the proof. \qed

3. Mean values of $u \ast v$

Theorem 1.1 is an immediate consequence of the following:

Theorem 3.1. If $0 < p < 1$ and $q \geq p$, then there is a constant $C = C_{p,q}$, such that

\begin{equation}
M_q \left( u \ast v, r^2 \right) \leq C (1 - r)^{1 - 1/p} A_p(u, r) A_q(v, r), \quad 0 < r < 1,
\end{equation}

for all $u, v \in h(\Delta)$. 
Proof. By Lemma 2.2, it suffices to prove that

\[ M_q \left( u \ast v, r^2 \right) \leq C (1 - r)^{1-1/p} M_p \left( u^+, r \right) M_q \left( v^+, r \right). \]

Let \( 0 < p < 1 \) and \( q \geq p \). For a fixed \( w \in T \) let

\[ F(r\zeta) = u \left( r\overline{\zeta} \right) v \left( rw\zeta \right), \quad 0 < r < 1, \quad \zeta \in T. \]

Then

\[ (u \ast v) \left( r^2 w \right) = (v \ast u) \left( r^2 w \right) = \int_T F(r\zeta) \, d\delta(\zeta) \]

(Here \( u \) and \( v \) are harmonic, but the proof of (3.2) is independent of this hypothesis.) By Lemma 2.1,

\[ \left| (u \ast v) \left( r^2 w \right) \right|^p \leq M^p_1(F, r) \]

\[ \leq C (1 - r)^{p-1} \int_T F^+(r\zeta)^p \, d\delta(\zeta). \]

Since obviously \( F^+(r\zeta) \leq u^+(r\overline{\zeta})v^+(rw\zeta) \), we get

\[ \left| (u \ast v) \left( r^2 w \right) \right|^p \leq C (1 - r)^{p-1} \int_T u^+ \left( r\overline{\zeta} \right)^p v^+ \left( rw\zeta \right)^p \, d\delta(\zeta). \]

Hence, by Minkowski’s inequality for \( M_s, s = q/p \geq 1 \),

\[ M^p_q \left( u \ast v, r^2 \right) = M_S \left( |u \ast v|^p, r^2 \right) \]

\[ \leq C (1 - r)^{p-1} \int_T \left[ \int_T u^+ \left( r\overline{\zeta} \right)^{ps} v^+ \left( rw\zeta \right)^{ps} \, d\delta(w) \right]^{1/s} \, d\delta(\zeta) \]

\[ = C (1 - r)^{p-1} \int_T u^+ \left( r\overline{\zeta} \right)^p \, d\delta(\zeta) \left[ \int_T v^+ \left( rw\zeta \right)^q \, d\delta(w) \right]^{p/q} \]

\[ = C (1 - r)^{p-1} M^p_p \left( u^+, r \right) M^p_p \left( v^+, r \right), \]

which was to be proved. \( \square \)

The following generalization of Theorem 1.1 follows directly from Theorem 3.1. Here

\[ h^p(\alpha) = \{ u \in h(\Delta) : M_p(u, r) = 0 \left( (1 - r)^{-\alpha} \right), r \to 1 \} \]

and

\[ h^0_0(\alpha) = \{ u \in h(\Delta) : M_p(u, r) = 0 \left( (1 - r)^{-\alpha} \right), r \to 1 \}, \]

where \( \alpha \) is a real number and \( 0 < p \leq \infty \). Thus \( h^p(0) = h^p \), while \( h^0_0(0) \) coincides with the space \( h^p_0 \) which was considered by Shapiro [5].
Theorem 3.2. Let $0 < p < 1$ and $q \geq p$. Then

(i) If $u \in h^p(\alpha)$ and $v \in h^q(\beta)$, then $u * v \in h^{\alpha + \beta + 1/p - 1}$;

(ii) If $u \in h^p_0(\alpha)$ (resp. $u \in h^p(\alpha)$) and $v \in h^q(\beta)$ (resp. $v \in h^q_0(\beta)$), then $u * v \in h^q_0(\alpha + \beta + 1/p - 1)$.

In contrast to the case of analytic functions, the spaces $h^p(\alpha)$ $(0 < p < 1)$ need not be trivial for $\alpha < 0$. For example, we have the following interesting fact.

Corollary 3.1. If $1/2 < p < 1$, then the space $h^p(1 - 1/p)$ is an infinite-dimensional algebra (relative to the operation $*$) with the Poisson kernel as unit.

Proof. That $h^p(1 - 1/p)$ is an algebra follows from Theorem 3.2(i). The rest follows from the fact that all the functions

$$P_w(z) = \frac{1 - |z|^2}{|1 - wz|^2}, \quad w \in T, \quad z \in \Delta,$$

(the rotates of the Poisson kernel $P = P_w$, $w = 1$) belong to $h^p(1 - 1/p)$. See [5] and [4].

REFERENCES

4. ——, Mean values of harmonic conjugates in the unit disc, Complex Variables 10 (1988), 53-65.