RESTRICTIONS OF OPEN MAPPINGS OF CONTINUA

WITOLD D. BULA AND E. D. TYMCHATYN

(Communicated by Dennis Burke)

Abstract. We give an example of an open map \( f : X \rightarrow Y \), where \( X \) is a locally connected one-dimensional continuum, such that there is a subcontinuum \( K \) of \( Y \) for which the restriction of \( f \) to a certain component of \( f^{-1}(K) \) is not open.

E. Duda had recently asked the first author whether the following question has a negative answer: Let \( f : X \rightarrow Y \) be an open mapping of a continuum \( X \) (i.e. a compact connected Hausdorff space) onto a Hausdorff space \( Y \). If \( L \) is a continuum in \( X \) such that \( L \) is a component of \( f^{-1}(f(L)) \), is \( f|L \), the restriction of \( f \) to \( L \), open? Duda’s question has a positive answer if the preimage of every continuum in \( X \) has only finitely many components. This is the case, for example, if the map \( f \) is finite-to-one, or, if the domain of \( f \) is hereditarily locally connected. Below (Theorem) we give a one-dimensional counterexample with a locally connected domain. We start by constructing an example with non-locally connected domain. It will serve to provide motivation for the more complicated main construction. Let us recall that a map \( f : X \rightarrow Y \) is open at \( x \in X \) if it carries neighborhoods of \( x \) to neighborhoods of \( f(x) \).

Example. Let \( v = (0, 1) \), \( a_0 = (0, 0) \) and \( a_n = (1/n, 0) \) for each positive integer \( n \). For each non-negative integer \( n \) let \( L_n \) be the line segment in the plane with endpoints \( v \) and \( a_n \). Then \( X = \bigcup \{L_n : n = 0, 1, 2, \ldots\} \), the harmonic fan, is the cone with vertex \( v \) over the convergent sequence \( \{a_0, a_1, a_2, \ldots\} \). Let \( B = \{[p_n, q_n] : n = 1, 2, \ldots\} \) be a basis of open intervals for the open interval \([0, 1]\). Define a map \( f_0 \) of the closed interval \([0, 1]\) to itself by letting \( f_0(0) = 0 \), \( f_0(1/3) = f_0(2/3) = 1/2 \), \( f_0(1) = 1 \) and extending \( f_0 \) linearly. Let \( \{f_n : n = 1, 2, \ldots\} \) be a sequence of homeomorphisms of \([0, 1]\) onto \([0, 1]\) such that for each \( n = 1, 2, \ldots\)

\[
\begin{align*}
(1) \quad & ||f_n - f_0|| < 1/n , \\
(2) \quad & f_n(p_n) < f_0(p_n) , \text{ and} \\
(3) \quad & f_0(q_n) < f_n(q_n) .
\end{align*}
\]

Received by the editors September 6, 1988 and, in revised from, April 20, 1989.
1980 Mathematics Subject Classification (1985 Revision). Primary 54C10.
Key words and phrases. Open mappings.
This research was supported in part by NSERC grant number A5616.

©1990 American Mathematical Society
0002-9939/90 $1.00 + .25 per page

233
Define \( f : X \to [0, 1] \) by \( f(x, y) = f_n(y) \) for \((x, y) \in L_n \). Then \( f \) is continuous and \( f \) is clearly open at every point of \( X \setminus \{0\} \times [1/3, 2/3] \). Let \( y \in [1/3, 2/3] \). Let \( B_y = \{ p_{n(i)} , q_{n(i)} \colon i = 1, 2, \ldots \} \subset B \) be a neighborhood base at \( y \) such that \( p_{n(i)} < p_{n(i+1)} < y < q_{n(i+1)} < q_{n(i)} \) for each \( i \). Then the sequence of open intervals \([0, 1] \setminus \{ p_{n(i)} \}, q_{n(i)} \setminus L_{n(i)} \) is eventually in every neighborhood of \((0, y)\). Hence, \( f(0, y) = f_0(y) \) is in the interior of \( f(U) \) for each neighborhood \( U \) of \((0, y)\) and \( f \) is open at \((0, y)\). Now \( K = \{0\} \times [0, 2/3] \) is a component of \( f^{-1} \circ f([0] \times [0, 2/3]) = f^{-1}([0, 1/2]) \) but \( f([0] \times [1/3, 2/3]) = \{1/2\} \) so \( f|K \) is not open.

A closed subset \( K \) of a space \( X \) is said to be a \( Z \)-set in \( X \) if for each \( \varepsilon > 0 \) there exists a mapping \( f : X \to X \setminus K \) such that \( f \) moves no point of \( X \) more than \( \varepsilon \) (see [2, p. 151]). We denote the closure [resp., the boundary] of a set \( A \) by \( \text{Cl}(A) \) [resp., \( \text{Bd}(A) \)].

**Theorem.** There exists an open map \( h : X \to N \) of a one-dimensional locally connected metric continuum \( X \) onto a Menger curve \( N \) such that there is a continuum \( K \) in \( X \) which is a component of \( f^{-1} \circ f(K) \) and such that \( f|K \) is not open.

**Proof.** Let \( M \) denote the Menger curve. Let \( K \) be an arc which is a \( Z \)-set in \( M \) and let \( A \subset K \) be an arc which does not contain an endpoint of \( K \). Let \( N = M/\text{Cl}(A) \) and let \( q : M \to N \) be the natural quotient map. Observe that \( N \) is homeomorphic to \( M \) by Anderson's characterization of the Menger curve ([1]; see also [3, Theorem 4.11]) and let \( a = q(A) \).

Let \( B = q(K) \). Then \( B \) is an arc in \( N \) which is a \( Z \)-set in \( N \) and such that \( a \) is not an endpoint of \( B \). Let \( \{x_n : n = 1, 2, \ldots \} \) be a dense set of non-endpoints of \( A \).

Let \( \{U_n : n = 1, 2, \ldots \} \) be a sequence of open neighborhoods of \( A \) such that \( M \supset \text{Cl}(U_1) \supset U_1 \supset \text{Cl}(U_2) \supset \cdots \supset \text{Cl}(U_n) \supset U_n \supset \text{Cl}(U_{n+1}) \supset \cdots \), \( M \neq \text{Cl}(U_1) \), \( \{U_n : n = 1, 2, \ldots \} = A \), each \( \text{Cl}(U_n) \) is a Menger curve and \( \text{Bd}(U_n) \) is a Cantor set which is a \( Z \)-set in both \( \text{Cl}(U_n) \) and \( M \setminus U_n \). Notice that \( q(\text{Bd}(U_n)) \) is a \( Z \)-set in both \( N \setminus q(U_n) \) and \( q(\text{Cl}(U_n)) \) for each \( n \).

We claim that there exists a sequence of homeomorphisms \( q_n : M \to N \) such that

1. \( \lim_{n \to \infty} q_n = q \),
2. \( q_n(K) = B \),
3. \( q_n(x_n) = a \), and
4. \( q_n|\text{M} \setminus U_n = q|\text{M} \setminus U_n \).

By Corollary 6.5 of [3], there exists for each \( n \) a homeomorphism \( q_n : M \to N \) such that \( q_n|\text{M} \setminus U_n = q|\text{M} \setminus U_n \), \( q_n(K) = B \) and \( q_n(x_n) = a \).

For each \( n = 1, 2, \ldots \) let \( A_n = \{x_{n,1}, \ldots, x_{n,m_n}\} \subset \text{M} \setminus K \) be a 1/n-mesh for \( M \). This is possible since \( K \) is nowhere dense in \( M \). For each \( i = 1, \ldots, m_n \), let \( U_{n,i} \) be an open neighborhood of \( x_{n,i} \) of diameter less
than \(1/n\) and such that \(CIU_{n,i} \cap (CIU_{n,j} \cup K) = \emptyset\) for \(i \neq j\). By choosing \(U_n\) to be sufficiently small we can also insure that \(CI(U_{n,i}) \cap CI(U_{n}) = \emptyset\).

Let \(X = M \times Z/\sim\), where \(Z\) is the convergent sequence \(\{0\} \cup \{1/n : n = 1, 2, \ldots\}\) and \(\sim\) is the smallest upper semi-continuous equivalence relation on \(M \times Z\) such that \((x, 1/n) \sim (x, 0)\) for \(x \in U_{n,i}\), where \(n = 1, 2, \ldots\) and \(i \in \{1, \ldots, m_n\}\). Let \(\pi_1 : M \times Z \overset{\text{onto}}{\longrightarrow} X\) be the natural quotient map. Then \(X\) is a continuum. We prove that \(X\) is locally connected. Let \(x \in X\). Since \(M\) is locally connected, let \(V\) be a small connected neighborhood of \(x\) in \(V\). Hence, \(A_m \cap V \neq \emptyset\) for each \(m \geq n\). For \(n\) large \(\pi_1(\bigcup \{V \times \{z\} | z \in Z \text{ and } z \leq 1/n \text{ or } \pi_1(x, z) = \pi_1(x, 0)\})\) is a small connected neighborhood of \(\pi_1(x, 0)\). If \(z \in Z\) is such that \(\pi_1(x, z) \neq \pi_1(x, 0)\) then \(\pi_1(V \times \{z\})\) is a small connected neighborhood of \(\pi_1(x, z)\). Define a mapping \(h : M \times Z \overset{\text{onto}}{\longrightarrow} N \times Z\) by \(h(x, 0) = q(0)\) and \(h(x, 1/n) = q_n\) for \(n = 1, 2, \ldots\). Let \(\sim_2\) be the smallest upper semi-continuous relation on \(N \times Z\) such that there is a commutative diagram of continuous maps and spaces

\[
\begin{array}{ccc}
M \times Z & \overset{h}{\longrightarrow} & N \times Z \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
X & \overset{F}{\longrightarrow} & N \times Z/\sim_2
\end{array}
\]

where \(\pi_2\) is the natural quotient map of \(N \times Z\) onto \(N \times Z/\sim_2\). There is a natural projection \(\pi : N \times Z/\sim_2 \overset{\text{onto}}{\longrightarrow} N\) such that \(\pi \circ \pi_2 : N \times Z \overset{\text{onto}}{\longrightarrow} N\) is the first coordinate projection. Let \(f = \pi \circ F : X \overset{\text{onto}}{\longrightarrow} N\). That \(f\) is open at points of \(X \setminus \pi_1(A \times \{0\})\) is clear. It follows from (6) and the openness of \(q_n\) that \(f\) is open at points of \(\pi_1(A \times \{0\})\). Clearly \(\pi_1(K \times \{0\})\) is a component of \(f^{-1}(B)\) and \(f|\pi_1(K \times \{0\})\) is not open.

References


Department of Mathematics, University of Saskatchewan, Saskatoon S7N 0W0 Canada