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BETTI NUMBERS FOR MODULES OF FINITE LENGTH

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ABSTRACT. Let R be a Gorenstein local ring of dimension $d < 5$ and let M be a module of finite length and finite projective dimension. If M is not isomorphic to R modulo a regular sequence, then the Betti numbers of M satisfy $\beta_i(M) > \binom{d}{i}$ for $0 < i < d$, and $\sum_{i=0}^{d} \beta_i(M) \geq 2^d + 2^{d-1}$.

Let $R, \mathfrak{m}, k$ be a regular local ring of dimension d and M be a module of finite length. Buchsbaum and Eisenbud [1] conjectured that the Betti numbers of M, $\beta_i(M)$, satisfy $\beta_i(M) \geq \binom{d}{i}$ for all i. Of course there is equality for all i if M is isomorphic to R modulo a maximal R-sequence. Santoni [18], extending ideas of Evans and Griffith [5], proved this conjecture for $R = k[[x_1, \dots, x_d]]$ and M a multigraded module. Charalambous [4] showed that in this case if M is not $k[[x_1, \dots, x_d]]$ mod a maximal multigraded sequence, then the $\beta_i(M)$ satisfy at least one of the following stronger inequalities: $\beta_i(M) \geq \binom{d}{i} + \binom{d-1}{i-1}$ for all i; or $\beta_i(M) \geq \binom{d}{i} + \binom{d-1}{i}$ for all i. This has as an immediate consequence that $\sum \beta_i(M) \geq 2^d + 2^{d-1}$ and that $\beta_i(M) > \binom{d}{i}$ for all i except possibly $i = 0$ or $i = d$. In this paper we prove the analogous results for modules of finite length and finite projective dimension over a local Cohen-Macaulay ring of dimension $d \leq 3$, or a local Gorenstein ring of dimension $d = 4$. We note that the Cohen-Macaulay assumption is essentially harmless since if a local ring has a module of finite length and finite projective dimension, then by P. Roberts' New Intersection Theorem [16] the ring must be Cohen-Macaulay. Linkage theory and its connection with the multiplicative structure of $\operatorname{Tor}_{\bullet}^{R}(R/I, k)$ provides a crucial tool in establishing constraints on the $\beta_i(M)$. See [15, 1, 13] for further details on these techniques. We also point out that Golod [7] has shown that the basic results of linkage theory remain valid for perfect ideals in a commutative noetherian ring.

Let $\mathscr{C}(R)$ be the set of isomorphism classes of modules of finite length and finite projective dimension which are not isomorphic to R modulo a maximal R-sequence. For each $[M] \in \mathscr{C}(R)$ we get a $d+1$ tuple of positive integers

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$\beta[M] = (\beta_0(M), \dots, \beta_d(M))$. We write $\beta[M] \ge \beta[N]$ if $\beta_i(M) \ge \beta_i(N)$ for all i. We begin with an easy observation.

Theorem 1. *For each Cohen-Macaulay local ring R of dimension d there is a finite set $[M_1], \dots, [M_n] \in \mathscr{C}(R)$ such that if $[M]$ is in $\mathscr{C}(R)$, then $\beta[M] \ge \beta[M_i]$ for some i.*

Proof. Let $I \subset k[x_0, \dots, x_d]$ be the ideal generated by $x^{\beta[M]} = \prod x_i^{\beta_i(M)}$ where $[M] \in \mathscr{C}(R)$. This ideal is generated by a finite subset of the generators. It is clear that the corresponding modules have the desired property. □

Now that we know that there are finitely many minimal $\beta[M]$ we can try to find them for each ring. We will use several well known theorems. We define I to be a Gorenstein ideal if it is perfect of grade g and $\beta_g(R/I) = 1$. It is easy to see that if I is Gorenstein and R/I has finite length, then $\beta_i(R/I) = \beta_{d-i}(R/I)$ for all i. Kunz [11] has shown that if I is a height g Gorenstein ideal then I cannot be minimally generated by $g+1$ elements. We check that his conclusion (and proof) still continue to hold in our setting in Lemma 2. It is well known (and elementary) that if M has finite length and finite projective dimension with $\beta[M] = (\beta_0(M), \dots, \beta_d(M))$ then $M^\vee = \operatorname{Ext}_R^d(M, R)$ also has finite length, finite projective dimension and $\beta[M^\vee] = (\beta_d(M), \dots, \beta_0(M))$. Furthermore, if R is Gorenstein, then $\dim_k \operatorname{socle} M = \beta_d(M)$ (combine Definition 1.20 and Theorem 6.10 of [10], for example). We also use the following result which can be extracted from Buchsbaum and Rim [3] or spelled out in Buchsbaum and Eisenbud [2, Corollary 3.2]: if A is an $n \times m$ matrix such that $I_n(A)$, the ideal generated by the n by n minors of A, has height $m-n+1$, then $\operatorname{coker} A$ is resolved by the "generic Buchsbaum-Rim complex". Consequently, if $\beta_0(M) = n$ and $\beta_1(M) = m$ for a module M of finite length, then $m \ge d+n-1$; and equality holds if and only if the remaining β_i's are given by the Buchsbaum-Rim complex, namely $\beta_i = \binom{n+i-3}{i-2}\binom{m}{n+i-1}$ for $i \ge 2$. Applying the same reasoning to the dual resolution of $M^\vee$, we obtain the following result.

Lemma 1. *If R is a Cohen-Macaulay local ring of dimension d and M is a module of finite length, then $\beta_1(M) - \beta_0(M) \ge d-1$ and $\beta_{d-1} - \beta_d \ge d-1$. If equality holds in one of these, then M or $M^\vee$ is resolved by the Buchsbaum-Rim complex.*

Lemma 2. *Let R be a Cohen-Macaulay local ring of dimension d and I be a Gorenstein ideal of height d. Then I cannot be minimally generated by $d+1$ elements.*

Proof. Following Kunz [11], we suppose that I is minimally generated by $d+1$ elements, and we may choose a minimal generating set $z_1, \dots, z_d, w$ with $\mathbf{z} = z_1, \dots z_d$ an R-sequence. Let $\mathbb{F}$ be a minimal free resolution of R/I, let $\mathbb{K}$ be the Koszul complex on $\mathbf{z}, w$, and let $\mathrm{H}_\bullet = \mathrm{H}_\bullet(\mathbb{K})$ be the Koszul homology of I. By Gulliksen [9, Proposition 1.4.9] it is enough to show that

H_1 is isomorphic to R/I, since then I would be generated by an R-sequence, in contradiction to our assumption. An easy calculation shows that H_1 is isomorphic to $((\mathbf{z}):w)/(\mathbf{z})$, the annihilator of w modulo $(\mathbf{z})$, and in turn this is isomorphic to $\mathrm{Ext}^d_R(R/I, R)$. (See, for example, Sally [17, Corollary 1.4.2] or observe that if $\alpha: \mathbb{K} \to \mathbb{F}$ is the map of complexes induced by the inclusion of ideals $(\mathbf{z}) \subset I$, then $((\mathbf{z}):w)/(\mathbf{z})$ is resolved by the mapping cone of the dual of α.) Since $\beta_d(R/I) = 1$, we conclude that H_1 is cyclic. On the other hand if we look at multiplication by w on $R/(\mathbf{z})$ we see that the kernel and cokernel have the same length. Thus $H_1 \cong ((\mathbf{z}):w)/(\mathbf{z})$ is a cyclic image of R/I with the same length as R/I, and hence, must be isomorphic to it. (Alternatively, we know that $\mathrm{Ext}^d_R(R/I, R) \cong R/I$ and direct application of Corollary 1.4.2 and Lemma 5.1.8 of [17] concludes the proof.) □

Theorem 2. *Let $R, \mathfrak{m}$ be a Cohen-Macaulay local ring of dimension $d < 4$ with $x_1, \dots, x_d$ a system of parameters. Then the minimal $\beta[M]$ and M_i are:*

$d = 1$	$(2, 2)$	$M = R/(x_1) \oplus R/(x_1)$
$d = 2$	$(1, 3, 2)$	$M = R/(x_1^2, x_1x_2, x_2^2)$
	$(2, 3, 1)$	$M = (R/(x_1^2, x_1x_2, x_2^2))^\vee$
$d = 3$	$(1, 4, 5, 2)$	$M = R/(x_1^2, x_1x_2, x_2^2, x_3)$
	$(2, 5, 4, 1)$	$M = (R/(x_1^2, x_1x_2, x_2^2, x_3))^\vee$
	$(1, 5, 5, 1)$	$M = R/(x_1^2, x_1x_3, x_2^2, x_2x_3, x_1x_2 + x_3^2)$
	$(2, 4, 4, 2)$	$M = \operatorname{coker}\begin{pmatrix} x_1 & x_2 & x_3 & 0 \\ 0 & x_1 & x_2 & x_3 \end{pmatrix}$

Proof. $d = 1$ is completely trivial.
$d = 2$ is obvious.
$d = 3$: Let $\beta[M]$ be a Betti sequence. By passing to $M^\vee$ if necessary we can assume $\beta_3(M) \geq \beta_0(M)$.

If $\beta_0(M) = 1$ then $\beta_1(M) \geq 4$; otherwise M is isomorphic to R modulo a maximal sequence. If $\beta_1(M) = 4$ then by Kunz's theorem $\beta_3(M) \geq 2$ and $\beta[M] \geq (1, 4, 5, 2)$. If $\beta_1(M) \geq 5$ then $\beta[M] \geq (1, 5, 5, 1)$.

If $\beta_0(M) \geq 2$ then $\beta_3(M) \geq 2$ and $\beta[M] \geq (2, 4, 4, 2)$ by Lemma 1.

The modules do have the desired Betti numbers. For $M = \frac{R}{(x_1^2, x_1x_2, x_2^2, x_3)}$ this follows from the fact that $J = (x_1^2, x_1x_2, x_2^2)$ is a 3-generated Cohen-Macaulay ideal and R/J has a resolution

$$\mathbb{F}: \qquad 0 \longrightarrow R^2 \longrightarrow R^3 \longrightarrow R \longrightarrow R/J \longrightarrow 0.$$

Obviously x_3 is regular on R/J and hence $M = R/(J, x_3)$ is resolved by $\mathbb{F} \otimes_R (R \xrightarrow{x_3} R)$ which yields the Betti sequence $\beta[M] = (1, 4, 5, 2)$.

The ideal $I = (x_1^2, x_1x_3, x_2^2, x_2x_3, x_1x_2 + x_3^2)$ is a standard example of a 5-generated height three Gorenstein ideal [1] and hence $\beta[R/I] = (1, 5, 5, 1)$.

Finally, for $M = \operatorname{coker}\begin{pmatrix} x_1 & x_2 & x_3 & 0 \\ 0 & x_1 & x_2 & x_3 \end{pmatrix}$ we notice that the ideal of 2×2 minors has height $3 = 4 - 2 + 1$. Thus M is resolved by the Buchsbaum-Rim complex and $\beta[M] = (2, 4, 4, 2)$. $\square$

The situation in dimension four is not as satisfactory. We produce a list of six sequences and show that every $\beta[M]$ is bigger than at least one of these. However we cannot produce the modules for two of these sequences (an M, $M^\vee$ pair). The first four sequences do come from modules by taking the four modules in dimension three and killing x_4. The new modules are resolved by tensoring the old resolution with $R \xrightarrow{x_4} R$.

The following results use techniques of linkage to eliminate the possibility that certain sequences with small $\sum \beta_i$ might occur. In the course of the arguments we appeal to a theorem of Kustin and Miller [13, Theorem 2.2] in which the possible algebra structures on $T_\bullet = \operatorname{Tor}_\bullet^R(R/I, k)$ are classified (see Kustin [12] for the same conclusion in case $\operatorname{char} k = 2$). As stated these results require us to assume that R, $\mathfrak{m}$ is Gorenstein. If $\operatorname{char} k \neq 2$, then by Grothendieck [8, 0_{III}10.3.1] R can be "inflated" so that its residue field has square roots; that is, there is a flat local homomorphism $R \to S$ so that $S/\mathfrak{m}S = K$ is an extension of k closed under square root. Now S is still Gorenstein of dimension four (see [14, Theorem 23.4]) and $\beta[S \otimes M] = \beta[M]$ for any R-module M of finite length. Hence we may as well replace R by S and assume that either $\operatorname{char} k = 2$ or k is closed under square roots.

Lemma 3. *Assume that R is a Gorenstein local ring and $\dim R = 4$. Then there is no module M of finite length with $\beta[M] = (1, 5, 8, r+4, r)$.*

Proof. If a counterexample M does exist then it has the form R/J where J is an almost complete intersection; hence $r \geq 2$ by Lemma 2. Choose a minimal generating set $(z_1, \ldots, z_4, w)$ for J with $\mathbf{z} = z_1, z_2, z_3, z_4$ a regular sequence, and let I be linked to J via $\mathbf{z}$. Let $\mathbb{G}$ be a minimal free resolution of R/J, and let $\mathbb{K}$ be the Koszul resolution of $R/(\mathbf{z})$. If $\gamma: \mathbb{K} \to \mathbb{G}$ is a map of complexes lifting $\gamma_0 = \mathrm{id}_R$, then R/I is resolved by the mapping cone of γ^*. It is easy to see that I is Gorenstein, and if $\bar{\gamma} = \gamma \otimes_R \mathrm{id}_k$, $t_i = \operatorname{rank} \bar{\gamma}_i$, then

$$\beta[R/I] = (1 - t_4,\ 4 - t_3 + r - t_4,\ 10 + r - t_2 - t_3,\ 8 - t_2,\ 1).$$

It follows that $t_4 = 0$, and $8 - t_2 = 4 + r - t_3 = \mu(I)$. By Kunz $\mu(I) = 5$ is impossible, so either $\mu(I) = 4$ or $\mu(I) \geq 6$.

We will exclude both cases by using the symmetry of linkage. Let $\mathbb{F}$ be a minimal free resolution of R/I. The minimal free resolution of $M = R/J$ is found by taking the mapping cone of α^* for any map of complexes $\alpha: \mathbb{K} \longrightarrow \mathbb{F}$ induced by $(\mathbf{z}) \subset I$, and then splitting off extraneous summands. We may assume that α is a map of DG-algebras, and in particular $\operatorname{im}(\bar{\alpha}_2) = (\operatorname{im} \bar{\alpha}_1)^2$ in T_1^2. Let $s_i = \operatorname{rank} \bar{\alpha}_i$.

If $\beta[R/I] = (1, 4, 6, 4, 1)$, then

$$\beta[M] = (1 - s_4, 1 - s_4 + 4 - s_3, 4 - s_3 + 6 - s_2, 6 - s_2 + 4 - s_1, 4 - s_1)$$

from which we conclude that $s_4 = 0$, $s_3 = 0$, $s_2 = 2$, $s_1 = 4 - r \le 2$. But $\bar{\alpha}_2$ only gets nonzero contributions from products of two distinct basis elements in the image of $\bar{\alpha}_1$. Hence $s_1 \le 2$ implies $s_2 \le \binom{2}{2}$, which contradicts $s_2 = 2$.

If $\beta_1(R/I) = \mu(I) \ge 6$, then $t_2 \le 2$ and

$$\beta[M] = (1-s_4, 5-s_4-s_3, 14-t_2-s_3-s_2, 14+r-t_2-t_3-s_2-s_1, 4+r-t_3-s_1),$$

from which we conclude that $s_4 = 0$, $s_3 = 0$, $s_2 = 6 - t_2 \ge 4$, and $s_1 = 4 - t_3 \le 4$. Since $s_2 \le \binom{s_1}{2}$ we must have $t_3 = 0$ and $s_1 = 4$. Let $\delta_1(I)$ denote the k-vector space dimension of T_1^2 in the DG-algebra $T_\bullet = \mathrm{Tor}_\bullet^R(R/I, k)$. This is the number of Koszul relations on the generators of I that are among the minimal relations. Since $s_1 = 4$ the sequence $\mathbf{z}$ begins a minimal set of generators of I, and anything that is nonzero in the image of $\bar{\alpha}_2$ will give a nonzero element in T_1^2. Hence $\delta_1(I) \ge s_2 \ge 4$. The structure of the DG-algebra $T_\bullet$ is given by [13, Theorem 2.2]. If $\delta_1(I) = p \ge 4$, then, since I is not a complete intersection, there is a basis $\{e_1, \dots, e_{\mu(I)}\}$ for F_1 so that $\{\bar{e}_{p+1}\bar{e}_i \mid 1 \le i \le p\}$ is a basis for T_1^2 and all other products $\bar{e}_i\bar{e}_j = 0$. But then $s_2 = 0$ if $d(e_{p+1})$ is not used in the linking sequence $\mathbf{z}$, and $s_2 \le 3$ if it is. In either case it is not possible to have $s_2 \ge 4$. □

Theorem 3. *Let R be a Gorenstein local ring of dimension four and $[M]$ be in $\mathscr{C}(R)$. Then $\beta[M]$ is bigger than at least one of the following*:

$$(1, 5; 9, 7, 2)\ (1, 6, 10, 6, 1)\ (2, 6, 8, 6, 2)\ (1, 6, 9, 6, 2)$$
$$(2, 7, 9, 5, 1)\qquad (2, 6, 9, 6, 1)$$

Proof. As before, by passing to $M^\vee$ if needed, we can assume that $\beta_0(M) \le \beta_4(M)$.

(i) If $\beta_0(M) = 1$ and $\beta_4(M) = 1$ then by Kunz's theorem $\beta_1 = \beta_3$ must be at least 6. Thus $\beta_2 \ge 10$ since $\sum(-1)^i\beta_i = 0$, and consequently $\beta[M] \ge (1, 6, 10, 6, 1)$.

(ii) If $\beta_0(M) = 1$ and $\beta_4(M) = 2$, we discuss the cases $\beta_1(M) = 5$ and $\beta_1(M) > 5$ separately.

(iia) If $\beta_1(M) = 5$ and M is a counterexample then $\beta[M]$ must be $(1, 5, 8, 6, 2)$ or $(1, 5, 7, 5, 2)$. The second possibility can be excluded by Lemma 1, since the Buchsbaum-Rim sequence for $M^\vee$ would have to be $(2, 5, 10, 10, 3)$. The first possibility is ruled out by Lemma 3. Thus $\beta[M] \ge (1, 5, 9, 7, 2)$.

(iib) If $\beta_0(M) = 1$, $\beta_1(M) > 5$ and $\beta_4(M) = 2$, any sequence that satisfies $\sum(-1)^i\beta_i = 0$ and $\beta_3 - \beta_4 \ge 3$ that is not bigger than $(1, 6, 9, 6, 2)$ would have $\beta_3 = 5$. Then by Lemma 1 the sequence for $M^\vee$ would be $(2, 5, 10, 10, 3)$, contradicting $\beta_4(M^\vee) = \beta_0(M) = 1$. Thus $\beta[M] \geq (1, 6, 9, 6, 2)$.

(iii) If $\beta_0(M) = 1$, $\beta_1(M) \geq 5$ and $r = \beta_4(M) \geq 3$ then the only possible sequences that are not bigger than any of the listed sequences would be $(1, 5, 7, r+3, r)$, $(1, 6, 8, r+3, r)$, or $(1, 5, 8, r+4, r)$. In each of the first two cases we may apply Lemma 1 to $M^\vee$ and so obtain $\beta_4(M^\vee) = \binom{r+1}{2} \geq 6$, a contradiction to $\beta_0(M) = 1$. The last case is ruled out by Lemma 3.

(iv) Finally if $\beta_4(M) \geq \beta_0(M) \geq 2$ then either the sequence is the Buchsbaum-Rim $(2, 5, 10, 10, 3)$, which is bigger than $(1, 5, 9, 7, 2)$; or it is bigger than $(2, 6, 8, 6, 2)$. □

We now obtain the desired corollary.

Corollary. *Let R be a Gorenstein local ring of dimension $d \leq 4$ and let $[M]$ be in $\mathscr{C}(R)$. Then*

$$\sum_{i=0}^{d} \beta_i(M) \geq 2^d + 2^{d-1}$$

and

$$\beta_i(M) > \binom{d}{i} \text{ for all } i \text{ such that } 0 < i < d.$$

Proof. We only have to check these claims for the minimal sequences given above, for which it is clear. □

This extends Charalambous' result to all Gorenstein local rings of dimension less than five. Since the Buchsbaum-Eisenbud conjecture is unknown in dimension five and is implied by the second of these results, the question of the extension to higher dimension is open; though it seems unlikely that a brute force combinatorial attack will work. Nevertheless it provides an interesting question. Another question is whether there exist modules giving the last two sequences in the dimension four case. If not, then the minimal modules in dimension four would arise as sections of those in dimension three. This would be rather intriguing.

Although we have been unable to decide if there exist finite length modules $M = R/I$ with $\beta[M] = (1, 6, 9, 6, 2)$ or not, we can show that such I cannot be generated by six quadrics. Note that in contrast for $\dim R = 3$ the minimal resolutions of cyclic modules that do not arise as hypersurfaces from dimension two, namely $\beta[M] = (1, 5, 5, 1)$, can be produced as $M = R/I$ with the ideal I generated by five quadrics.

Theorem 4. *Let $R = K[[x_1, \ldots, x_4]]$ with K a field. Let $M = R/I$ be a finite length module with I generated by six quadrics, $f_1, \ldots, f_6$. Then $\beta[M]$ cannot be $(1, 6, 9, 6, 2)$.*

Proof. We recall that the Hilbert function $H(M)$ is given by $H(M)(n) = \dim_K M_n$ where M_n denotes the n^{th} graded piece of M. Since our modules are of finite length we can give the Hilbert function as a finite list of nonzero values. We can assume that $\mathbf{f} = f_1, \ldots, f_4$ forms an R-sequence. Then $H(R/(\mathbf{f})) = (1, 4, 6, 4, 1)$. If $s \in R_4$ generates the socle of $R/(\mathbf{f})$ (which we know is 1-dimensional since $R/(\mathbf{f})$ is a complete intersection), then there must be a quadric r so that $rf_5 \equiv s \bmod (\mathbf{f})$. It follows that s is in I, so the socle of R/I lives in degree at most three. (This can also be seen by considering a comparison map of the minimal homogeneous resolutions of $R/(\mathbf{f})$ and R/I. The last twist in the former is 8, so the last twist in the latter must be no more than 7.) Consequently $H(R/I) = (1, 4, 4, h_3)$. Now $\dim_K \operatorname{socle}(R/I) = \beta_4(R/I) = 2$; hence $\operatorname{socle}(R/I)$ cannot be all of $(R/I)_2$, and therefore h_3 cannot be zero. By the same token $h_3 \leq 2$. We handle the cases $h_3 = 1$ and $h_3 = 2$ separately.

Case 1. The Hilbert function of R/I is $(1, 4, 4, 1)$. This Hilbert function can exist as an ideal generated by six quadrics, as the standard example of a height four Gorenstein ideal $I = (x_1^2, x_2^2 + x_3x_4, x_3^2, x_4^2, x_2x_3, x_2x_4)$ illustrates. By looking at the minimal graded resolution of R/I, or by direct calculation, one can see that

$$\begin{aligned} H(R/I) = H(R) - 6H(R[-2]) + 5H(R[-3]) + 5H(R[-4]) \\ - 6H(R[-5]) + H(R[-7]), \end{aligned}$$

so that $H(R/I)(n) = \binom{n+3}{3} - 6\binom{n+1}{3} + 5\binom{n}{3} + 5\binom{n-1}{3} - 6\binom{n-2}{3} + \binom{n-4}{3}$. As usual $R[-a]$ denotes the free module R with shifted grading: $R[-a]_n = R_{n-a}$. It follows that any minimal homogeneous resolution of a graded module with Hilbert function $(1, 4, 4, 1)$ must have one copy of R, five copies of $R[-3]$, five copies of $R[-4]$, and one copy of $R[-7]$ in the even positions of the resolution, while it must have six copies of $R[-2]$ and six copies of $R[-5]$ in the odd positions. Any additional terms would consist of the same number of copies of $R[-a]$ added to both the odd and even parts of the resolution. Also there are nonzero maps $R[-b] \to R[-a]$ only if $b \geq a$. Thus the only possible resolutions using just the absolutely required terms have $\beta[R/I] = (1, 6, 10, 6, 1)$. Since these betti numbers already add to 24, as do the desired betti numbers (1, 6, 9, 6, 2), there is no resolution of 6 quadrics with Hilbert function $(1, 4, 4, 1)$ and the desired betti numbers.

Case 2. The Hilbert function of R/I is (1, 4, 4, 2). We link I to J via the R-sequence $\mathbf{f}$. Then $H(R/J) = (1, 2, 2)$ since $(1, 4, 4, 2, 0) + (0, 0, 2, 2, 1) = (1, 4, 6, 4, 1)$ by the duality relation for Hilbert functions of linked algebras; see [6, Theorem 3.1] or compute the Hilbert function of R/J from its resolution as a mapping cone. The link J is generated by $\mathbf{f}$ together with generators coming from the map $\alpha_4 \colon R \to R^2$ induced by a comparison map α from the Koszul resolution of $R/(\mathbf{f})$ to a minimal resolution of R/I. Since J has two linearly independent linear terms these must arise from that map. Thus J is generated by two linearly independent linear elements

ℓ_1, ℓ_2 and $\mathbf{f}$, from which it follows that J has an R-sequence $\mathbf{z}$ consisting of ℓ_1, ℓ_2 and two quadrics q_1, q_2. The Hilbert function of $R/(\mathbf{z})$ is $(1, 2, 1)$. Since R/J is a homomorphic image of $R/(\mathbf{z})$, the Hilbert function can only get smaller termwise, and hence cannot be $(1, 2, 2)$. $\square$

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