WHEN IS A FLAT ALGEBRA OF FINITE TYPE?

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Abstract. Let $A$ denote a commutative Noetherian domain. For an intermediate ring $A \subseteq B \subseteq A_x$ flat over $A$, it is shown that $B$ is an $A$-algebra of finite type. This is followed by an intrinsic description of the flatness of $B$ over $A$ and the asymptotic behavior of certain prime divisors. As an application, flat ideal-transforms are characterized.

Let $A$ denote a commutative Noetherian domain. Let $Q(A)$ be the quotient field of $A$. By an intermediate ring we understand a commutative ring $B$ with $A \subseteq B \subseteq Q(A)$. In his paper [R], Richman proved that an intermediate ring $B$ is a Noetherian ring provided $B$ is flat over $A$. In general $B$ is not an $A$-algebra of finite type. For instance, take $A$ the integers and $B$ the rationals. If $B$ is an intermediate ring of finite type over $A$, then $B \subseteq A_x$ for a certain $0 \neq x \in A$. This necessary condition is also sufficient for an $A$-flat intermediate ring $B$ being of finite type over $A$ as shown in the following

Theorem 1. Let $B$ be an intermediate ring with $A \subseteq B \subseteq A_x$, $x \neq 0$. Suppose that $B$ is flat as an $A$-module. Then $B$ is an $A$-algebra of finite type.

Here $A_x$ denotes the localization of $A$ with respect to $\{x^n : n \in \mathbb{N}\}$. To prove Theorem 1 we need a few preliminaries about certain asymptotic prime divisors and flatness.

Proposition. Let $B$ be an intermediate ring with $A \subseteq B \subseteq A_x$. Then

$$\text{Ass}_A(x^n B \cap A/x^n A), \ n \geq 1,$$

forms an increasing set of prime ideals stabilizing for large $n$ to a finite set equal to $\text{Ass}_A(B/A)$.

Proof. First we will show that the considered sets of associated prime ideals are increasing. This follows easily by virtue of the monomorphism

$$x^n B \cap A/x^n A \to x^{n+1} B \cap A/x^{n+1} A, \quad n \geq 1,$$
defined by \( r + x^n A \mapsto rx + x^{n+1} A \). Because of

\[
\text{Ass}_A(x^n B \cap A/x^n A) \subseteq \text{Ass}_A A/x^n A = \text{Ass}_A A/x A,
\]
ote that \( x \) is a nonzero divisor, the desired sets stabilize for large \( n \) to a finite set.

Now let \( P \in \text{Ass}_A(x^n B \cap A/x^n A) \). Then there exists an element \( r \in x^n B \cap A/x^n A \) such that \( P = x^n A : r \). Put \( q = r/x^n \in B \). It follows that \( q \notin A \) and \( P = A : q \), as can be easily seen. That is, \( P \in \text{Ass}_A(B/A) \). In order to prove the reverse inclusion let \( P \in \text{Ass}_A(B/A) \). Then there is an element \( q = r/x^n \in B \setminus A, r \in A, \) such that \( P = A : q \). As above, \( P = x^n A : r \) and \( r \in x^n B \cap A/x^n A \). That is, \( P \in \text{Ass}_A(x^n B \cap A/x^n A) \), now, by the previous consideration the claim follows. \( \square \)

As an additional feature to our investigations we will characterize when an intermediate ring \( A \subseteq B \subseteq A_x \) is flat over \( A \). This is done in

**Theorem 2.** An intermediate ring \( B \) with \( A \subseteq B \subseteq A_x, \ x \neq 0, \) is flat as an \( A \)-module if and only if \( B = PB \) for all \( P \in \text{Ass}_A(B/A) \).

**Proof.** Suppose \( B \) is flat as an \( A \)-module. Let \( PB \) be a proper ideal for some \( P \in \text{Ass}_A(B/A) \). By the Going Down theorem, see [M], there exists a prime ideal \( Q \) minimal over \( PB \) such that \( P = Q \cap A \). The induced homomorphism \( A_P \to B_Q \) makes \( B_Q \) into a faithful flat \( A_P \)-module. Because of \( B_Q \subseteq Q(A_P) \) it yields \( B_Q = A_P \) and \( A_P = B_P \), contracting the choice of \( P \) as an element of \( \text{Ass}_A(B/A) \). Now suppose that \( B = PB \) for all prime ideals \( P \in \text{Ass}_A(B/A) \). To show that \( B \) is flat over \( A \), it is enough to prove that \( B_Q \) is flat over \( A_P \) for all \( Q \in \text{Spec} B \) and \( P = Q \cap A \), see [M, (3.J)]. Let \( Q \in \text{Spec} B \). Then \( P = Q \cap A \notin \text{Ass}_A(B/A) \) because \( Q \supseteq PB \) is a proper ideal. If \( P \notin \text{Supp}_A(B/A) \), then \( A_P = B_P = B_Q \) and \( B_Q \) is flat over \( A_P \). Suppose \( P \in \text{Supp}_A(B/A) \). Then there is a prime ideal \( P' \in \text{Ass}_A(B/A) \) such that \( P' \subseteq P \). Note that \( \text{Supp}_A(B/A) \) and \( \text{Ass}_A(B/A) \) have the same set of minimal prime ideals, see [B, §1, 3, Corollary 1]. But then

\[
B = P'B \subseteq PB \subseteq Q \subseteq B,
\]
contracting the choice of \( Q \). \( \square \)

With the previous notations, set

\[
I = x^n A : (x^n B \cap A),
\]
where \( n \) is chosen such that \( \text{Ass}_A(x^n B \cap A/x^n A) \) stabilizes. Note that \( I = \text{Ann}_A(x^n B \cap A/x^n A) \).

**Corollary 1.** An intermediate ring \( B \) as above is flat as an \( A \)-module if and only if \( B = IB \).

**Proof.** Because \( \text{Ass}_A(B/A) = \text{Ass}_A(x^n B \cap A/x^n A) \) it follows, see [B, §1, 4, Theorem 2], that \( I \) and \( \text{Ass}_A(B/A) \) have the same set of minimal prime ideals. Then the claim follows by the theorem. \( \square \)
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For an ideal $I$ of $A$ let

$$T(I) = \{ q \in \mathbb{Q}(A) : I^n q \subseteq A \text{ for some } n \in \mathbb{N} \}$$

denote the ideal-transform of $A$ with respect to $I$. Note that $T(I)$ is an intermediate ring with $A \subseteq T(I) \subseteq A_x$, $0 \neq x \in I$.

**Corollary 2.** The ideal-transform $T(I)$ is flat over $A$ if and only if $T(I) = JT(I)$, where $J$ denotes the intersection of those primary components of a reduced primary decomposition of $xA$, $0 \neq x \in I$, for which the associated prime ideals contain $I$.

**Proof.** First note that

$$xT(I) \cap A = xA: \langle I \rangle = \bigcup_{k \geq 1} xA: I^k.$$

Furthermore $\text{Ass}_A(x^nT(I) \cap A/x^nA) = \text{Ass} A/xA \cap V(I)$ for all $n \geq 1$. As can be easily seen, $J = xA: (xA: \langle I \rangle)$. Because $J$ and $\text{Ann}_A(x^nT(I) \cap A/x^nA)$ have the same radical, Corollary 1 proves the claim. □

The previous corollary generalizes part of [E, (3.2)]. For a further investigation about the flatness of ideal-transforms see also [S].

**Corollary 3.** With the notations of Corollary 1, an intermediate ring $B$ is flat over $A$ if and only if $T(I) = B$ and $\text{Spec} A \setminus V(I)$ is an affine scheme.

**Proof.** It holds in general that if $I$ is an ideal of a Noetherian domain $A$, then the quasi-affine scheme $\text{Spec} A \setminus V(I)$ is affine if and only if $T(I) = IT(I)$, see [H]. By Corollary 1 this proves the “only if” part of the statement. Assume that $B$ is flat over $A$, i.e., $B = IB$ by Corollary 1. It is enough to prove that $T(I) = B$. First note that $T(I) \subseteq T(IB) = B$. On the other side let $r/x^n \in B$. Then $r \in x^nB \cap A$. Because

$$\text{Supp}_A(x^nB \cap A/x^nA) \subseteq V(I)$$

there is an integer $m$ that $I^m r \subseteq x^nA$, i.e., $r/x^n \in T(I)$ as required. □

In particular, let $J$ be an ideal of a Noetherian domain $A$ such that $\text{Spec} A \setminus V(J)$ is an affine scheme. Then $T(J)$ is flat over $A$ and an $A$-algebra of finite type by Theorem 1.

**Proof of Theorem 1.** Let $I = x^nA: (x^nB \cap A)$ as defined above. By [B], $\text{Ass}_A(B/A)$ and $\text{Supp}_A(B/A)$ have the same set of minimal prime ideals containing $I$. Therefore $\text{Supp}_A(B/A) \subseteq V(I)$ because of $I = \text{Ann}_A(x^nB \cap A/x^nA)$ and the proposition. By Corollary 1, $B = IB$, i.e., there is a decomposition of the unit

$$1 = \sum_{i=1}^s r_i q_i, \quad r_i \in I, \quad q_i \in B, \quad i = 1, \ldots, s.$$

Now we claim that $B = A[q_1, \ldots, q_s]$. Clearly $B \supseteq A[q_1, \ldots, q_s]$. Let $y \in B$. Because of $\text{Supp}_A(B/A) \subseteq V(I)$ there is an integer $n$ such that $I^n y \subseteq A$. The
\( n \)th power of the above relation yields
\[
1 = \sum_{|a|=n} r_a q^a, \quad a = (a_1, \ldots, a_s),
\]
with \( r_a \in I^n \). Therefore
\[
y = \sum_{|a|=n} (r_a y) q^a \in A[q_1, \ldots, q_s],
\]
as required. \( \Box \)

References


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