PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 109, Number 2, June 1990

WHEN IS A FLAT ALGEBRA OF FINITE TYPE?

PETER SCHENZEL

(Communicated by Louis J. Ratliff, Jr.)

ABSTRACT. Let A denote a commutative Noetherian domain. For an intermediate ring $A \subseteq B \subseteq A_x$ flat over A, it is shown that B is an A-algebra of finite type. This is followed by an intrinsic description of the flatness of B over A and the asymptotic behavior of certain prime divisors. As an application, flat ideal-transforms are characterized.

Let A denote a commutative Noetherian domain. Let $\mathbb{Q}(A)$ be the quotient field of A. By an intermediate ring we understand a commutative ring B with $A \subseteq B \subseteq \mathbb{Q}(A)$. In his paper [R], Richman proved that an intermediate ring B is a Noetherian ring provided B is flat over A. In general B is not an A-algebra of finite type. For instance, take A the integers and B the rationals. If B is an intermediate ring of finite type over A, then $B \subseteq A_x$ for a certain $0 \neq x \in A$. This necessary condition is also sufficient for an A-flat intermediate ring B being of finite type over A as shown in the following

Theorem 1. *Let B be an intermediate ring with $A \subseteq B \subseteq A_x$, $x \neq 0$. Suppose that B is flat as an A-module. Then B is an A-algebra of finite type.*

Here A_x denotes the localization of A with respect to $\{x^n : n \in \mathbb{N}\}$. To prove Theorem 1 we need a few preliminaries about certain asymptotic prime divisors and flatness.

Proposition. *Let B be an intermediate ring with $A \subseteq B \subseteq A_x$. Then*

$$\operatorname{Ass}_A(x^n B \cap A / x^n A),\ n \geq 1,$$

forms an increasing set of prime ideals stabilizing for large n to a finite set equal to $\operatorname{Ass}_A(B/A)$.

Proof. First we will show that the considered sets of associated prime ideals are increasing. This follows easily by virtue of the monomorphism

$$x^n B \cap A / x^n A \to x^{n+1} B \cap A / x^{n+1} A, \qquad n \geq 1,$$

Received by the editors March 23, 1989.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 13C11, 13E05; Secondary 14A15, 14B25.

Key words and phrases. Intermediate ring, flat algebra, algebra of finite type, prime divisor, ideal-transform, affine scheme.

defined by $r + x^n A \mapsto rx + x^{n+1} A$. Because of

$$\mathrm{Ass}_A(x^n B \cap A/x^n A) \subseteq \mathrm{Ass}_A A/x^n A = \mathrm{Ass}_A A/xA,$$

note that x is a nonzero divisor, the desired sets stabilize for large n to a finite set.

Now let $P \in \mathrm{Ass}_A(x^n B \cap A/x^n A)$. Then there exists an element $r \in x^n B \cap A \backslash x^n A$ such that $P = x^n A : r$. Put $q = r/x^n \in B$. It follows that $q \notin A$ and $P = A : q$, as can be easily seen. That is, $P \in \mathrm{Ass}_A(B/A)$. In order to prove the reverse inclusion let $P \in \mathrm{Ass}_A(B/A)$. Then there is an element $q = r/x^n \in B \backslash A$, $r \in A$, such that $P = A : q$. As above, $P = x^n A : r$ and $r \in x^n B \cap A \backslash x^n A$. That is, $P \in \mathrm{Ass}_A(x^n B \cap A/x^n A)$, now, by the previous consideration the claim follows. □

As an additional feature to our investigations we will characterize when an intermediate ring $A \subseteq B \subseteq A_x$ is flat over A. This is done in

Theorem 2. *An intermediate ring B with $A \subseteq B \subseteq A_x$, $x \neq 0$, is flat as an A-module if and only if $B = PB$ for all $P \in \mathrm{Ass}_A(B/A)$.*

Proof. Suppose B is flat as an A-module. Let PB be a proper ideal for some $P \in \mathrm{Ass}_A(B/A)$. By the Going Down theorem, see [M], there exists a prime ideal Q minimal over PB such that $P = Q \cap A$. The induced homomorphism $A_P \to B_Q$ makes B_Q into a faithful flat A_P-module. Because of $B_Q \subseteq \mathbb{Q}(A_P)$ it yields $B_Q = A_P$ and $A_P = B_P$, contracting the choice of P as an element of $\mathrm{Ass}_A(B/A)$. Now suppose that $B = PB$ for all prime ideals $P \in \mathrm{Ass}_A(B/A)$. To show that B is flat over A, it is enough to prove that B_Q is flat over A_P for all $Q \in \mathrm{Spec}\, B$ and $P = Q \cap A$, see [M, (3.J)]. Let $Q \in \mathrm{Spec}\, B$. Then $P = Q \cap A \notin \mathrm{Ass}_A(B/A)$ because $Q \supseteq PB$ is a proper ideal. If $P \notin \mathrm{Supp}_A(B/A)$, then $A_P = B_P = B_Q$ and B_Q is flat over A_P. Suppose $P \in \mathrm{Supp}_A(B/A)$. Then there is a prime ideal $P' \in \mathrm{Ass}_A(B/A)$ such that $P' \subseteq P$. Note that $\mathrm{Supp}_A(B/A)$ and $\mathrm{Ass}_A(B/A)$ have the same set of minimal prime ideals, see [B, §1, 3, Corollary 1]. But then

$$B = P'B \subseteq PB \subseteq Q \subseteq B,$$

contracting the choice of Q. □

With the previous notations, set

$$I = x^n A : (x^n B \cap A),$$

where n is chosen such that $\mathrm{Ass}_A(x^n B \cap A/x^n A)$ stabilizes. Note that $I = \mathrm{Ann}_A(x^n B \cap A/x^n A)$.

Corollary 1. *An intermediate ring B as above is flat as an A-module if and only if $B = IB$.*

Proof. Because $\mathrm{Ass}_A(B/A) = \mathrm{Ass}_A(x^n B \cap A/x^n A)$ it follows, see [B, §1, 4, Theorem 2], that I and $\mathrm{Ass}_A(B/A)$ have the same set of minimal prime ideals. Then the claim follows by the theorem. □

For an ideal I of A let

$$T(I) = \{q \in \mathbb{Q}(A) \colon I^n q \subseteq A \text{ for some } n \in \mathbb{N}\}$$

denote the ideal-transform of A with respect to I. Note that $T(I)$ is an intermediate ring with $A \subseteq T(I) \subseteq A_x$, $0 \neq x \in I$.

Corollary 2. *The ideal-transform $T(I)$ is flat over A if and only if $T(I) = JT(I)$, where J denotes the intersection of those primary components of a reduced primary decomposition of xA, $0 \neq x \in I$, for which the associated prime ideals contain I.*

Proof. First note that

$$xT(I) \cap A = xA \colon \langle I \rangle = \bigcup_{k \geq 1} xA \colon I^k .$$

Furthermore $\operatorname{Ass}_A(x^n T(I) \cap A / x^n A) = \operatorname{Ass} A/xA \cap V(I)$ for all $n \geq 1$. As can be easily seen, $J = xA \colon (xA \colon \langle I \rangle)$. Because J and $\operatorname{Ann}_A(x^n T(I) \cap A / x^n A)$ have the same radical, Corollary 1 proves the claim. □

The previous corollary generalizes part of [E, (3.2)]. For a further investigation about the flatness of ideal-transforms see also [S].

Corollary 3. *With the notations of Corollary 1, an intermediate ring B is flat over A if and only if $T(I) = B$ and $\operatorname{Spec} A \backslash V(I)$ is an affine scheme.*

Proof. It holds in general that if I is an ideal of a Noetherian domain A, then the quasi-affine scheme $\operatorname{Spec} A \backslash V(I)$ is affine if and only if $T(I) = IT(I)$, see [H]. By Corollary 1 this proves the "only if" part of the statement. Assume that B is flat over A, i.e., $B = IB$ by Corollary 1. It is enough to prove that $T(I) = B$. First note that $T(I) \subseteq T(IB) = B$. On the other side let $r/x^n \in B$. Then $r \in x^n B \cap A$. Because

$$\operatorname{Supp}_A(x^n B \cap A / x^n A) \subseteq V(I)$$

there is an integer m that $I^m r \subseteq x^n A$, i.e., $r/x^n \in T(I)$ as required. □

In particular, let J be an ideal of a Noetherian domain A such that Spec $A \backslash V(J)$ is an affine scheme. Then $T(J)$ is flat over A and an A-algebra of finite type by Theorem 1.

Proof of Theorem 1. Let $I = x^n A \colon (x^n B \cap A)$ as defined above. By [B], $\operatorname{Ass}_A(B/A)$ and $\operatorname{Supp}_A(B/A)$ have the same set of minimal prime ideals containing I. Therefore $\operatorname{Supp}_A(B/A) \subseteq V(I)$ because of $I = \operatorname{Ann}_A(x^n B \cap A / x^n A)$ and the proposition. By Corollary 1, $B = IB$, i.e., there is a decomposition of the unit

$$1 = \sum_{i=1}^{s} r_i q_i , \quad r_i \in I , \quad q_i \in B , \qquad i = 1, \dots, s .$$

Now we claim that $B = A[q_1, \dots, q_s]$. Clearly $B \supseteq A[q_1, \dots, q_s]$. Let $y \in B$. Because of $\operatorname{Supp}_A(B/A) \subseteq V(I)$ there is an integer n such that $I^n y \subseteq A$. The

nth power of the above relation yields

$$1 = \sum_{|a|=n} r_a q^a, \qquad a = (a_1, \ldots, a_s),$$

with $r_a \in I^n$. Therefore

$$y = \sum_{|a|=n} (r_a y) q^a \in A[q_1, \ldots, q_s],$$

as required. □

References

[B] N. Bourbaki, *Algèbre commutative*, Hermann, Paris, 1961, Chap. 4.

[E] P. M. Eakin, Jr., W. Heinzer, D. Katz, and L. J. Ratliff, Jr., *Note on ideal-transforms, Rees rings, and Krull rings*, J. Algebra **110** (1987), 407–419.

[H] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, Heidelberg, and Berlin, 1977.

[M] H. Matsumura, *Commutative algebra*, 2nd ed., Benjamin, New York, 1980.

[R] F. Richman, *Generalized quotient rings*, Proc. Amer. Math. Soc. **16** (1965), 794–799.

[S] P. Schenzel, *Flatness and ideal-transforms of finite type*, (to appear).

Sektion Mathematik der Martin-Luther-Universität Halle-Wittenberg, Postfach, Halle, DDR-4010, German Democratic Republic