ON ORDERED Λ-BOUNDED VARIATION

FRANCISZEK PRUS-WIŚNIOWSKI

(Communicated by R. Daniel Mauldin)

Abstract. An example is given of a continuous real function that is of ordered Λ-bounded variation but not of Λ-bounded variation. No special assumptions on Λ are required.

Preliminaries and motivation. Let Λ = (λ_i) be a Λ-sequence, i.e. a nonincreasing convergent-to-0 sequence of positive numbers with \( \sum \lambda_i = \infty \). For \( m = 0, 1, 2, \ldots \) we will denote the Λ-sequence \( (\lambda_{m+j})_{j=1}^{\infty} \) by \( \Lambda_m \). Let \( \langle \alpha, \beta \rangle \) be a closed interval and \( x \) be a real function defined on \( \langle \alpha, \beta \rangle \). For a subinterval \( I = \langle s, t \rangle \) of \( \langle \alpha, \beta \rangle \) we set \( r(x, I) = x(t) - x(s), \) \( x(I) = |x(t) - x(s)| \) and \( \Lambda \vartheta(x; \langle s, t \rangle) = \sup \sum \lambda_i x(I_i) \), where the supremum is taken over all sequences \( (I_i) \) of nonoverlapping closed subintervals of \( \langle s, t \rangle \). If \( \Lambda - \vartheta(x; \langle \alpha, \beta \rangle) \) is finite then we say \( x \) is of Λ-bounded variation (on \( \langle \alpha, \beta \rangle \)) and write \( x \in \Lambda BV \).

A finite collection \( \{I_i; i = 1, \ldots, n\} \) of nonoverlapping closed subintervals of \( \langle s, t \rangle \) will be called left-ordered (right-ordered) if \( \sup I_{i+1} \leq \inf I_i (\sup I_i \leq \inf I_{i+1}) \). Set \( \Lambda - \vartheta_l(x; \langle s, t \rangle) = \sup \sum \lambda_i x(I_i) \) and \( \Lambda - \vartheta_r(x; \langle s, t \rangle) = \sup \sum \lambda_i x(I_i) \), where the supremum \( \sup (\sup) \) is taken over all left-ordered (right-ordered) finite collections of nonoverlapping subintervals of \( \langle s, t \rangle \). If \( \Lambda - \vartheta_l(x; \langle \alpha, \beta \rangle) = \max \{\Lambda - \vartheta_l(x; \langle \alpha, \beta \rangle), \) \( \Lambda - \vartheta_r(x; \langle \alpha, \beta \rangle)\} \) is finite, then we say \( x \) is of ordered Λ-bounded variation and write \( x \in O\Lambda BV \).

It is easy to see that the above definition is equivalent to the classical one ([4], p. 75).

Clearly, \( \Lambda BV \subset O\Lambda BV \). In [4] D. Waterman asked: Is this inclusion proper? C. L. Belna constructed in [1] a continuous function \( f \in OHBV \) such that \( f \notin HBV \) for the Λ-sequence \( H = (1/i) \). Since the method of C. L. Belna is applicable only to \( OHBV \) and not to \( O\Lambda BV \) defined in the obvious way, D. Waterman has repeated his question in [6] and in the present paper we give the positive answer in the general case.

Received by the editors August 11, 1988 and, in revised form, July 6, 1989.
1980 Mathematics Subject Classification (1985 Revision). Primary 26A45.
Key words and phrases. Function of Λ-bounded variation, function of ordered Λ-bounded variation.
Results. Let $n, k_1, \ldots, k_n$ be positive integers and $(a_i)_{i=1}^n$ be a finite decreasing sequence of positive numbers. For a function $x: (\alpha, \beta) \to \mathbb{R}$ we will write $x \in F_{k_1, \ldots, k_n}^{a_1, \ldots, a_n}(\alpha, \beta)$ if there exists a finite decreasing sequence $(t_i)_{i=1}^{r_n+1}$ of points of $(\alpha, \beta)$ such that $t_1 = \beta$, $t_{r_n+1} = \alpha$ and

$$
x(t_i) = 0 \quad \text{for } i = 1, 3, 5, \ldots, r_n + 1;
$$
$$
x(t_i) = a_j \quad \text{for } j = 1, \ldots, n \quad i = r_{j-1} + 2, r_{j-1} + 4, \ldots, r_j;
$$
$$
x \text{ is linear on } (t_{i+1}, t_i) \text{ for } i = 1, \ldots, r_n,
$$

where $r_j = 2 \sum_{i=1}^j k_i$ for $j = 0, 1, \ldots, n$.

Lemma 1. Let $A = (\lambda_j)$ be a $A$-sequence and $x \in F_{k_1, \ldots, k_n}^{a_1, \ldots, a_n}(\alpha, \beta)$. Then

$$
\Lambda - \var(x; (\alpha, \beta)) = \Lambda - \operatorname{lovar}(x; (\alpha, \beta)) = \sum_{j=1}^n a_j \sum_{i=r_{j-1}+1}^{r_j} \lambda_i.
$$

Proof. Setting $P_i = (t_{i+1}, t_i)$ for $i = 1, \ldots, r_n$, we have $\sup P_{i+1} = \inf P_i$. Thus

$$
\Lambda - \var(x; (\alpha, \beta)) \geq \Lambda - \operatorname{lovar}(x; (\alpha, \beta)) \geq \sum_{i=1}^{r_n} \lambda_i x(P_i) = \sum_{j=1}^n a_j \sum_{i=r_{j-1}+1}^{r_j} \lambda_i.
$$

Hence, it is enough to show that

$$
\Lambda - \var(x; (\alpha, \beta)) \leq \sum_{i=1}^{r_n} \lambda_i x(P_i).
$$

Observe that for every finite collection $\{I_i: i = 1, \ldots, s\}$ of nonoverlapping subintervals of $(\alpha, \beta)$ there exists a collection $\{I'_i: i = 1, \ldots, s\}$ of subintervals of $(\alpha, \beta)$ such that $I'_i \subset I_i$, $x(I'_i) = x(I_i)$ and $\{t_i: i = 1, \ldots, r_n+1\} \cap \operatorname{int} I'_j = \emptyset$ for $j = 1, \ldots, s$. Without loss of generality we may assume that the sets $Z_i = \{j = 1, \ldots, s: I'_i \subset (t_{i+1}, t_i)\}$ are nonempty for $i = 1, \ldots, r_n$. Setting $m_i = \min Z_i$ and arranging all $m_i$ in the decreasing order $w_1 > \cdots > w_{r_n}$, we have $\lambda_{w_i} \leq \lambda_i$ for $i = 1, \ldots, r_n$. Therefore, by ([2], Theorem 368)

$$
\sum_{j=1}^s \lambda_j x(I_j) = \sum_{i=1}^{r_n} \sum_{j \in Z_i} \lambda_j x(I'_j) \leq \sum_{i=1}^{r_n} \lambda_{m_i} \sum_{j \in Z_i} x(I'_j) \leq \sum_{i=1}^{r_n} \lambda_{m_i} x(P_i) \leq \sum_{i=1}^{r_n} \lambda_{w_i} x(P_i) \leq \sum_{i=1}^{r_n} \lambda_i x(P_i).
$$

So by ([3], Theorem 1) the proof is complete.  \(\Box\)
For a \( \Lambda \)-sequence \( \Lambda = (\lambda_i) \), a function \( x \in F_{k_1, \ldots, k_n}(\alpha, \beta) \) and \( s = 1, \ldots, r_n \) we set \( j_s = \max\{j = 1, \ldots, n: r_{j-1} < s\} \), \( p_s = s - r_{j_s-1} \) and

\[
v(\Lambda, x, s) = a_j \sum_{i=1}^{p_s} \lambda_i + \sum_{m=1}^{j_s-1} a_m \sum_{i=p_s+r_{j_s-1}-r_{m-1}}^{p_s+r_{j_s-1}} \lambda_i.
\]

**Lemma 2.** Let \( \Lambda = (\lambda_i) \) be a \( \Lambda \)-sequence and \( x \in F_{k_1, \ldots, k_n}(\alpha, \beta) \). Then \( \Lambda - \text{rovar}(x: (\alpha, \beta)) = \max\{v(\Lambda, x, s): s = 1, \ldots, r_n\} \).

**Proof.** Set \( P_i = (t_{i+1}, t_i) \) for \( i = 1, \ldots, r_n \). Since

\[
\Lambda - \text{rovar}(x; (\alpha, \beta)) \geq \max\{v(\Lambda, x, s): s = 1, \ldots, r_n\}.
\]

it is enough to show that

\[
\Lambda - \text{rovar}(x; (\alpha, \beta)) \leq \max\{v(\Lambda, x, s): s = 1, \ldots, r_n\}.
\]

Let \( \{I_j: j = 1, \ldots, k\} \) be a finite right-ordered collection of nonoverlapping subintervals of \( (\alpha, \beta) \). Without loss of generality we may assume that \( \{I_i: i = 1, \ldots, r_n + 1\} \cap \text{int} I_j = \emptyset \) for \( j = 1, \ldots, k \). Let \( Z_i = \{j = 1, \ldots, k: I_j \subset (t_{i+1}, t_i)\} \) for \( i = 1, \ldots, r_n \). Setting \( L = \{i = 1, \ldots, r_n: Z_i \neq \emptyset\} \) and \( c = \text{card} L \), we have \( c \leq r_n \). Further, set \( m_i = \min Z_i \) for \( i \in L \). Then \( \lambda_j \leq \lambda_{m_i} \) for \( j \in Z_i \). Arranging all elements of \( L \) in the decreasing order \( w_1 > \cdots > w_c \), we have \( m_{w_i} \geq i \) and \( w_i \geq c + 1 - i \) for \( i = 1, \ldots, c \). Thus \( \lambda_{m_{w_i}} \leq \lambda_i \) and \( x(P_{w_i}) \leq x(P_{c+1-i}) \) for \( i = 1, \ldots, c \). Finally,

\[
\sum_{j=1}^{k} \lambda_j x(I_j) = \sum_{i=1}^{c} \sum_{j \in Z_{w_i}} \lambda_j x(I_j) \leq \sum_{i=1}^{c} \lambda_{m_{w_i}} \sum_{j \in Z_{w_i}} x(I_j)
\]

\[
\leq \sum_{i=1}^{c} \lambda_{m_{w_i}} x(P_{w_i}) \leq \sum_{i=1}^{c} \lambda_i x(P_{c+1-i}) = v(\Lambda, x, c) \leq \max\{v(\Lambda, x, s): s = 1, \ldots, r_n\}.
\]

**Lemma 3.** Let \( n \) be a positive integer. Then for arbitrary numbers \( a, \gamma, \varepsilon > 0 \) and arbitrary \( \Lambda \)-sequence \( \Lambda = (\lambda_i) \) there exist positive numbers \( a_1 > \cdots > a_n \) with \( a_1 \leq a \) and positive integers \( k_1, \ldots, k_n \) such that \( \Lambda - \text{var}(x; (\alpha, \beta)) = n\gamma \) and \( \Lambda - \text{rovar}(x; (\alpha, \beta)) \leq \gamma + \varepsilon \) for \( x \in F_{k_1, \ldots, k_n}(\alpha, \beta) \).

**Proof.** Let \( n = 1 \). Given positive numbers \( a, \gamma, \varepsilon \) and a \( \Lambda \)-sequence \( \Lambda = (\lambda_i) \) we set

\[
k_1 = \min \left\{ m = 1, 2, \ldots: a \sum_{i=1}^{2m} \lambda_i \geq \gamma \right\} \quad \text{and} \quad a_1 = \gamma \left( \sum_{i=1}^{k_1} \lambda_i \right)^{-1}.
\]
For \( x \in F_{k_1}^{a_1}(\alpha, \beta) \) we have by Lemma 1

\[
\Lambda - \text{var}(x; (\alpha, \beta)) = a_1 \sum_{i=1}^{2k_1} \lambda_i = \gamma
\]

and by Lemma 2

\[
\Lambda - \text{rovar}(x; (\alpha, \beta)) \leq \max\left\{ a_1 \sum_{i=1}^{s} \lambda_i : s = 1, \ldots, 2k_1 \right\} = \gamma < \gamma + \varepsilon.
\]

Now assume that \( n \) is a positive integer such that for every positive number \( a, \gamma, \varepsilon \) and every \( \Lambda \)-sequence \( \Lambda \) there exist positive numbers \( a_1 > \cdots > a_n \) with \( a_1 \leq a \) and positive integers \( k_1, \ldots, k_n \) such that \( \Lambda - \text{var}(x; (\alpha, \beta)) = n\gamma \) and \( \Lambda - \text{rovar}(x; (\alpha, \beta)) \leq \gamma + \varepsilon \) for \( x \in F_{k_1}^{a_1}, \ldots, F_{k_n}^{a_n}(\alpha, \beta) \). It is easy to see that if \( x \in F_{k_1}^{a_1}, \ldots, F_{k_n}^{a_n}(\alpha, \beta) \) then \( x \circ h \in F_{k_1}^{a_1}, \ldots, F_{k_n}^{a_n}(\alpha', \beta) \), where \( \alpha' = (\alpha + \beta)/2 \) and \( h(t) = 2t - \beta \) for \( t \in (\alpha', \beta) \). Hence, given positive numbers \( a, \gamma, \varepsilon \) and a \( \Lambda \)-sequence \( \Lambda = (\lambda_i) \), there exist positive numbers \( a_1 > \cdots > a_n \) with \( a_1 \leq a \) and positive integers \( k_1, \ldots, k_n \) such that \( \Lambda - \text{var}(x; (\alpha', \beta)) = n\gamma \) and \( \Lambda - \text{rovar}(x; (\alpha', \beta)) \leq \gamma + \varepsilon/2 \) for \( x \in F_{k_1}^{a_1}, \ldots, F_{k_n}^{a_n}(\alpha', \beta) \). Set \( k = \sum_{i=1}^{n} k_i \) and

\[
(+) \quad i_0 = \min\left\{ i = 1, 2, \ldots : \frac{\lambda_{2k+i}}{\lambda_{2k}} \leq \frac{\varepsilon}{4\gamma + 2\varepsilon} \right\},
\]

\[
(++) \quad d = \min\left\{ \frac{a_n}{2}, \varepsilon \left( 4 \sum_{i=1}^{2k} \lambda_i \right)^{-1}, \varepsilon \left( 4 \sum_{i=2k+1}^{2k+i_0} \lambda_i \right)^{-1} \right\},
\]

\[
k_{n+1} = \min\left\{ m = 1, 2, \ldots : \sum_{i=2k+1}^{2k+2m} \lambda_i \geq \frac{\gamma}{d} \right\},
\]

\[
a_{n+1} = \gamma \left( \sum_{i=2k+1}^{2k+k_{n+1}} \lambda_i \right)^{-1}.
\]

Then \( a_{n+1} \leq d < a_n \) and \( \Lambda_{(2k)} - \text{var}(x; (\alpha, \alpha')) = \gamma \) for \( x \in F_{k_{n+1}}^{a_{n+1}}(\alpha, \alpha') \).

Given functions \( x_1 \in F_{k_1}^{a_1}, \ldots, F_{k_n}^{a_n}(\alpha', \beta) \), \( x_2 \in F_{k_{n+1}}^{a_{n+1}}(\alpha, \alpha') \), we set

\[
y(t) = \begin{cases} 
x_1(t) & \text{for } t \in (\alpha', \beta), 
x_2(t) & \text{for } t \in (\alpha, \alpha').
\end{cases}
\]
Then $y \in F^{a_1, \ldots, a_{n+1}}_{k_1, \ldots, k_{n+1}}(\alpha, \beta)$ and by Lemma 1

\[
\Lambda - \text{var}(y; (\alpha, \beta)) = \sum_{j=1}^{n+1} a_j \sum_{i=r_{j-1}+1}^{r_j} \lambda_i
= a_{n+1} \sum_{i=2k+1}^{2k+2k_{n+1}} \lambda_i + \sum_{j=1}^{n} a_j \sum_{i=r_{j-1}+1}^{r_j} \lambda_i
= \Lambda_{(2k)} - \text{var}(x_2; (\alpha, \alpha')) + \Lambda - \text{var}(x_1; (\alpha', \beta))
= (n+1)\gamma.
\]

For $s = 1, \ldots, r_n$ we have

(a) if $1 \leq s \leq 2k = r_n$ then

\[
v(\Lambda, y, s) = v(\Lambda, x_1, s) \leq \Lambda - \text{rovar}(x_1; (\alpha', \beta)) \leq \gamma + \varepsilon/2;
\]

(b) if $2k + 1 \leq s$ then

(b1) if $p_s \leq 2k + i_0$ then by $(++)$ and by (a)

\[
v(\Lambda, y, s) = a_{n+1} \sum_{i=1}^{p_s} \lambda_i + \sum_{m=1}^{n} a_m \sum_{i=p_s+r_{n-1}} \lambda_i
\leq a_{n+1} \sum_{i=1}^{2k} \lambda_i + a_{n+1} \sum_{i=2k+1}^{2k+i_0} \lambda_i + \sum_{m=1}^{n} a_m \sum_{i=r_{n-1}+1} \lambda_{p_s+i}
\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \sum_{m=1}^{n} a_m \sum_{i=r_{n-1}+1} \lambda_i
= \frac{\varepsilon}{2} + v(\Lambda, y, s) \leq \gamma + \varepsilon;\]

(b2) if $p_s > 2k + i_0$ then by $(+)$

\[
\frac{\lambda_{p_s+i}}{\lambda_1} \leq \frac{\lambda_{2k+i_0}}{\lambda_{2k}} \leq \frac{\varepsilon}{4\gamma + 2\varepsilon} \quad \text{for } i = 1, \ldots, 2k
\]

and thus, by (a),

\[
v(\Lambda, y, s) = a_{n+1} \sum_{i=1}^{2k} \lambda_i + a_{n+1} \sum_{i=2k+1}^{2k+i_0} \lambda_i + \sum_{m=1}^{n} a_m \sum_{i=r_{n-1}+1} \lambda_{p_s+i}
\leq \frac{\varepsilon}{4} + a_{n+1} \sum_{i=2k+1}^{2k+2k_{n+1}} \lambda_i + \sum_{m=1}^{n} a_m \sum_{i=r_{n-1}+1} \lambda_{2k+i_0+i}
\leq \frac{\varepsilon}{4} + \Lambda_{(2k)} - \text{var}(x_2; (\alpha, \alpha')) + \sum_{m=1}^{n} a_m \sum_{i=r_{n-1}+1} \frac{\lambda_i \varepsilon}{4\gamma + 2\varepsilon}
= \frac{\varepsilon}{4} + \gamma + \frac{\varepsilon}{4\gamma + 2\varepsilon} v(\Lambda, y, 2k).
\]
Therefore, by Lemma 2, \( \Lambda - \text{rovar}(y; (\alpha, \beta)) \leq \gamma + \varepsilon \). Since for every \( x \in F_{k_1, \ldots, k_{n+1}}(\alpha, \beta) \) we have \( \Lambda - \text{var}(x; (\alpha, \beta)) = \Lambda - \text{var}(y; (\alpha, \beta)) \) and \( \Lambda - \text{rovar}(x; (\alpha, \beta)) = \Lambda - \text{rovar}(y; (\alpha, \beta)) \), the proof is complete. \( \square \)

**Theorem.** For every \( \Lambda \)-sequence \( \Lambda = (\lambda_i) \) there exists a continuous \( x \in OABV - \Lambda BV \).

**Proof.** Let \( (t_n) \) be an increasing convergent-to-\( \beta \) sequence of points of \( (\alpha, \beta) \) with \( t_1 = \alpha \) and let \( \sum \varepsilon_i \) be a convergent series of positive numbers. By Lemmas 3 and 1 for \( n = 1, 2, \ldots \) there exist positive numbers \( a_1^n > \cdots > a_{2^n}^n \) with

\[
a_i^n \leq d_n,
\]
positive integers \( k_1^n, \ldots, k_{2^n}^n \) and functions \( x_n \in F_{k_1^n, \ldots, k_{2^n}^n}(t_n, t_{n+1}) \) such that

\[
\begin{align*}
\lambda_{(w_{n-1})} - \text{var}(x_n; (t_n, t_{n+1})) &= \lambda_{(w_{n-1})} - \text{lovar}(x_n; (t_n, t_{n+1})) = 2^n2^{-n} \\
\lambda_{(w_{n-1})} - \text{rovar}(x_n; (t_n, t_{n+1})) &\leq 2^{-n} + \varepsilon_n/2
\end{align*}
\]
for \( n = 1, 2, \ldots \), where

\[
\begin{align*}
d_1 &= 1 \quad \text{and for } n = 1, 2, \ldots \\
d_{n+1} &= \min \left\{ \frac{a_i^n}{2}, \varepsilon_{n+1} \left( 2 \sum_{i=1}^{n} \lambda_i \right)^{-1}, \varepsilon_{n+1} \left( 2 \sum_{i=w_n+1}^{n} \lambda_i \right)^{-1} \right\}; \\
w_n &= 2 \sum_{j=1}^{n} \sum_{i=1}^{2^j} k_i^j \quad \text{for } n = 0, 1, 2, \ldots; \\
v_{n+1} &= \min \left\{ i = 1, 2, \ldots: 4 \sum_{j=1}^{n} \sum_{i=1}^{2^j} k_i^j a_i^j < \frac{\varepsilon_{n+1}}{\lambda_{w_n+i}} \right\} \quad \text{for } n = 1, 2, \ldots.
\end{align*}
\]

For \( n = 1, 2, \ldots \) we define a continuous function \( y_n: (t_1, t_{n+1}) \to \mathbb{R} \) by \( y_n(t) = x_i(t) \) for \( t \in (t_i, t_{i+1}) \), \( i = 1, \ldots, n \). We shall show that for \( n = 1, 2, \ldots \) the following inequality holds:

\[
\Lambda - \text{lovar}(y_n; (t_1, t_{n+1})) \leq 1 + \sum_{i=1}^{n} \varepsilon_i.
\]

By (2) it is clear for \( n = 1 \), because \( x_1 = y_1 \). Assume that \( n \) is a positive integer such that (7) holds. Let \( \{I_i: i = 1, \ldots, l\} \) be a finite left-ordered collection of nonoverlapping subintervals of \( (t_1, t_{n+2}) \). Without loss of generality we may assume \( t_{n+1} \notin \text{int} \ I_i \) for \( i = 1, \ldots, l \). Set \( z = \max\{i = 1, \ldots, l: I_i \subset (t_{n+1}, t_{n+2})\} \) and \( h = \min\{z, w_n\} \). If \( z \leq w_n + v_{n+1} \) then we have by the
inductional assumption and by (1), (4)

\[
\sum_{i=1}^{n} \lambda_i y_{n+1}(I_i) \leq \sum_{i=1}^{h} \lambda_i \xi_{n+1}(I_i) + \sum_{i=w_n+1}^{z} \lambda_i \xi_{n+1}(I_i) + \sum_{i=z+1}^{l} \lambda_i y_n(I_i)
\]

\[
\leq \sum_{i=1}^{h} \lambda_i a_{n+1}^i + \sum_{i=w_n+1}^{z} \lambda_i a_{n+1}^i + \sum_{i=z+1}^{l} \lambda_i y_n(I_i)
\]

\[
\leq \frac{\varepsilon_{n+1}}{2} + \frac{\varepsilon_{n+1}}{2} + \Lambda - \text{lovar}(y_n; \langle t_1, t_{n+1} \rangle)
\]

\[
\leq 1 + \sum_{i=1}^{n+1} \varepsilon_i.
\]

If \( z > w_n + v_{n+1} \) then by (6)

\[
\sum_{i=1+z}^{l} \lambda_i y_{n+1}(I_i) \leq \lambda_{z+1} \sum_{i=z+1}^{l} y_n(I_i)
\]

\[
\leq 2\lambda_{w_n+v_{n+1}} \sum_{j=1}^{2^n} \sum_{i=1}^{k_j} a_j^i \leq \frac{\varepsilon_{n+1}}{2}
\]

and by (1), (4), (2)

\[
\sum_{i=1}^{z} \lambda_i y_{n+1}(I_i) \leq \sum_{i=1}^{w_n} \lambda_i \xi_{n+1}(I_i) + \sum_{i=w_n+1}^{z} \lambda_i \xi_{n+1}(I_i)
\]

\[
\leq a_{n+1}^i \sum_{i=1}^{w_n} \lambda_i + \Lambda(\omega_n) - \text{var}(x_{n+1}; \langle t_{n+1}, t_{n+2} \rangle)
\]

\[
\leq \frac{\varepsilon_{n+1}}{2} + 1.
\]

Therefore,

\[
\Lambda - \text{lovar}(y_{n+1}; \langle t_1, t_{n+2} \rangle) \leq 1 + \sum_{i=1}^{n+1} \varepsilon_i.
\]

Now, we shall show that

\[
(8) \quad \Lambda - \text{rovar}(y_n; \langle t_1, t_{n+1} \rangle) \leq \sum_{i=1}^{n} 2^{-i} + \sum_{i=1}^{n} \varepsilon_i
\]

for \( n = 1, 2, \ldots \). For \( n = 1 \) it is clear by (2). Assume that \( n \) is a positive integer such that (8) holds. Let \( \{I_i: i = 1, \ldots, l\} \) be a finite right-ordered collection of nonoverlapping subintervals of \( \langle t_1, t_{n+2} \rangle \). Without loss of generality we may assume that \( t_{n+1} \notin \text{int} I_i \) for \( i = 1, \ldots, l \). Setting \( z = \max\{i = 1, \ldots, l: I_i \subset \langle t_1, t_{n+1} \rangle\} \) (max \( \emptyset = 0 \)) and \( h = \min\{l, \max\{x, w_n\}\}, \)
we have by (1), (4), (3) and by the inductive assumption

\[ \sum_{i=1}^{l} \lambda_i y_{n+1}(I_i) \]

\[ \leq \sum_{i=1}^{z} \lambda_i y_i(n, I_i) + \sum_{i=z+1}^{h} \lambda_i x_{n+1}(I_i) + \sum_{i=h+1}^{l} \lambda_i x_{n+1}(I_i) \]

\[ \leq \Lambda - \text{rovar}(y_n; (t_1, t_{n+1})) + a_{n+1}^{\varnothing} \sum_{i=1}^{w} \lambda_i + \Lambda_{(w_n)} - \text{var}(x_{n+1}; (t_{n+1}, t_{n+2})) \]

\[ \leq \sum_{i=1}^{n} 2^{-i} + \sum_{i=1}^{n} \epsilon_i + \frac{\epsilon_{n+1}}{2} + 2^{-n-1} + \frac{\epsilon_{n+1}}{2} . \]

Therefore,

\[ \Lambda - \text{rovar}(y_{n+1}; (t_1, t_{n+2})) \leq \sum_{i=1}^{n} 2^{-i} + \sum_{i=1}^{n} \epsilon_i . \]

Finally, we define the function \( y: (\alpha, \beta) \to \mathbb{R} \) by \( y(\beta) = 0 \) and \( y(t) = x_n(t) \) for \( t \in (t_n, t_{n+1}) \), \( n = 1, 2, \ldots \). The function \( y \) is continuous, since \( a_1^{\varnothing} \leq 2^{-n+1} \) for \( n = 1, 2, \ldots \). It is easy to see that \( \Lambda - \text{var}(y; (\alpha, \beta)) \geq \sum_{n} \Lambda_{(w_n)} - \text{var}(x_n; (t_n, t_{n+1})) \). Thus, by (2) \( x \) is not of \( \Lambda \)-bounded variation.

Observe that for every finite left-ordered collection \( \{I_i: i = 1, \ldots, l\} \) of nonoverlapping subintervals of \( (\alpha, \beta) \) there exists a positive integer \( n \) and a finite left-ordered collection \( \{I'_i: i = 1, \ldots, l\} \) such that \( I'_i \subset I_i \), \( x(I'_i) = x(I_i) \) and \( \sup I'_i \leq t_{n+1} \). Thus

\[ \sum_{i=1}^{l} \lambda_i y(I_i) \leq \Lambda - \text{lovar}(y_n; (t_1, t_{n+1})) \]

and by (7) we have

\[ \Lambda - \text{lovar}(y; (\alpha, \beta)) \leq 1 + \sum_{i} \epsilon_i . \]

It may be proved in a similar manner that

\[ \Lambda - \text{rovar}(y; (\alpha, \beta)) \leq 1 + \sum_{i} \epsilon_i . \]

This fact together with (9) implies that \( y \in OABV \). \( \square \)

The remarks from §§2 and 3 of [5] are true for arbitrary \( \Lambda \)-sequence \( \Lambda \). In particular, \( OABV \) is a Banach space with norm

\[ \|g\| = |g(\alpha)| + \Lambda - \text{ovar}(g; (\alpha, \beta)) \]
and one has

**Corollary.** \( \Lambda BV \) is a first category subset of \( OA BV \) for arbitrary \( \Lambda \)-sequence \( \Lambda \).

Observe that the knowledge of construction of \( x \in OA BV - \Lambda BV \) is not necessary to prove the above corollary. Namely, let \( x \in OA BV - \Lambda BV \). Then there exists a sequence \( \{I_n\} \) of nonoverlapping subintervals of \( (\alpha, \beta) \) such that \( \sum \lambda_n x(I_n) = \infty \). It follows that there exists a subsequence \( \{I_{n_i}\} \) such that \( r(x, I_{n_i}) > 0 \) and \( \sum \lambda_i r(x, I_{n_i}) = \sum \lambda_i x(I_{n_i}) = \infty \) (we replace \( x \) by \( -x \), if necessary). Set

\[
m_k = \min \left\{ m = 1, 2, \ldots : \sum_{i=1}^{m} \lambda_i r(x, I_{n_i}) \geq k \right\}
\]

and

\[
F_k(g) = \sum_{i=1}^{m_k} \lambda_i r(g, I_{n_i})
\]

for \( k = 1, 2, \ldots ; \ g \in OA BV \). It is easy to see that for \( k = 1, 2, \ldots \), \( F_k \) is a continuous linear functional on \( OA BV \). Using D. Waterman's arguments ([5], proof of Proposition 3), we complete the proof.

**References**


**Institute of Mathematics, University of Szczecin, Ul. Wielkopolska 15, 70-451 Szczecin, Poland**