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RELATIONS BETWEEN BANACH FUNCTION ALGEBRAS AND THEIR UNIFORM CLOSURES

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ABSTRACT. Let A be a Banach function algebra on a compact Hausdorff space X. In this paper we consider some relations between the maximal ideal space, the Shilov boundary and the Choquet boundary of A and its uniform closure $\overline{A}$. As an application we determine the maximal ideal space, the Shilov boundary and the Choquet boundary of algebras of infinitely differentiable functions which were introduced by Dales and Davie in 1973.

For some notations, definitions, elementary and known results, one can refer to [2] and [3].

Let X be a compact Hausdorff space and let $C(X)$ denote the space of all continuous complex valued functions on X. A function algebra on X is a subalgebra of $C(X)$ which contains the constants and separates the points of X. If there is an algebra norm on A so that $\|f\cdot g\| \le \|f\|\cdot\|g\|$ for all $f, g \in A$, then A is called a normed function algebra. A complete normed function algebra on X is called a Banach function algebra on X. If the norm of a Banach function algebra is the uniform norm on X; i.e. $\|f\|_X = \sup_{x\in X} |f(x)|$, it is called a uniform algebra on X. If A is a function algebra on X, then $\overline{A}$, the uniform closure of A, is a uniform algebra on X.

If $(A, \|\cdot\|)$ is a Banach function algebra on X, for every $x \in X$ the map $\phi_x : A \to \mathbf{C}$, defined by $\phi_x(f) = f(x)$, is a nonzero continuous complex homomorphism on A and so $\phi_x \in M_A$, where M_A is the maximal ideal space of A. We call ϕ_x the evaluation homomorphism at x. Clearly for every $f \in A$,

$$\|f\|_X = \sup_{x\in X} |f(x)| = \sup_{x\in X} |\phi_x(f)| \le \sup_{\phi\in M_A} |\phi(f)| = \|\hat{f}\|_{M_A} \le \|f\|,$$

where $\hat{f}$ is the Gelfand transform of f.

The Banach function algebra A on X is called natural, if every $\phi \in M_A$ is given by an evaluation homomorphism ϕ_x at some $x \in X$; or, in other

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words, the map $x \to \phi_x$ is surjective. In fact M_A and X are homeomorphic through this mapping whenever A is natural. When A is natural we usually write $M_A \cong X$.

The Shilov boundary of A (with respect to X) is the smallest closed boundary of A (with respect to X) and it is denoted by $\Gamma(A, X)$.

If A is a function algebra on X and A^* is the conjugate (dual) space of $(A, \|\cdot\|_X)$, then the state space of A, denoted by $K(A)$, is defined by $K(A) = \{\phi \in A^* : \|\phi\| = \phi(1) = 1\}$. $K(A)$ is a weak *-compact Hausdorff convex subset of the closed unit ball in A^*. The Choquet boundary of A (with respect to X) is the set of all $x \in X$ for which ϕ_x is an extreme point of $K(A)$ and it is denoted by $\mathrm{Ch}(A, X)$. It is known that $\overline{\mathrm{Ch}}(A, X) = \Gamma(A, X)$.

Now we require a number of definitions and results which we now state briefly. For further details one can refer to [1].

Let X be a perfect compact plane set. A complex-valued function f on X is differentiable at z_0 if

$$f'(z_0) = \lim_{\substack{z \to z_0 \\ (z \in X)}} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

The algebra of functions on X with continuous kth derivative is denoted by $D^k(X)$. $D^\infty(X)$ is the algebra of functions with derivatives of all orders. Let $\{M_k\}_{k=0}^{\infty}$ be a sequence of positive numbers such that $M_0 = 1$ and for every $k \geq 1$,

$$(*) \qquad \frac{M_k}{M_r \cdot M_{k-r}} \geq \binom{k}{r} \qquad (r = 0, 1, \dots, k).$$

We define

$$D(X, \{M_k\}) = \left\{ f \in D^\infty(X) : \sum_{k=0}^{\infty} \frac{\|f^{(k)}\|_X}{M_k} < \infty \right\}$$

and a norm on $D(X, \{M_k\})$ by

$$\|f\|_D = \sum_{k=0}^{\infty} \frac{\|f^{(k)}\|_X}{M_k}.$$

Taking $M_r = r!$ $(r = 0, 1, \dots, k)$ and $1/M_r = 0$ $(r = k+1, \dots)$, we regard $D^k(X)$ as being an algebra of the type $D(X, \{M_k\})$. Hence for $f \in D^k$,

$$\|f\|_{D^k} = \sum_{r=0}^{k} \frac{\|f^{(r)}\|_X}{r!}.$$

Clearly $(D^k(X), \|\cdot\|_{D^k})$ and $(D(X, \{M_k\}), \|\cdot\|_D)$ are normed function algebras on X.

A compact plane set X is called uniformly regular if it is rectifiably arcwise connected and if the geodesic metric is uniformly equivalent to the Euclidean metric on X. In other words, there exists a constant C such that for every $z, w \in X$, $\delta(z, w) \leq C|z - w|$, where $\delta(z, w)$ is the geodesic distance between z and w. When X is uniformly regular $D^k(X)$ and $D(X, \{M_k\})$ are Banach function algebras on X with the appropriate norm [1]. If D is any of the preceding algebras then the subalgebra of D which is generated by the coordinate function z is denoted by D_P and the subalgebra generated by the rational functions with poles off X that belong to D is denoted by D_R. Clearly D_P and D_R are Banach function algebras on X and $D_P \subseteq D_R \subseteq D \subseteq D^k(X)$ for each k $(k = 1, 2, \ldots)$. Moreover, according to Lemma (1.5) in [1], $D^1(X) \subseteq R(X)$, where $R(X)$ is the uniform closure of rational functions on the uniformly regular set X with poles off X.

Let $P_0 = 1$ and $P_k = \sqrt[k]{M_k/k!}$ for $k \geq 1$ (we take $P_k = \infty$ if $M_k = \infty$). It is easy to see that if P_k is bounded, none of the preceding algebras are natural for any X. We are thus interested in the case when P_k is unbounded. We include the algebra $D^k(X)$ under this head. It is easy to show that $D_R(X, \{M_k\})$ is natural when $P_k \to \infty$ as $k \to \infty$ (see [1]).

It was also proved in [1] that $M_{D_P} \cong \widehat{X}$, where $\widehat{X}$ is the polynomial convex hull of X. We shall give another proof for this result which is much simpler than that in [1].

In this paper X always stands for a compact Hausdorff space. If A is a function algebra on X then it is easy to see that the restriction map $F \colon K(\overline{A}) \to K(A)$ defined by $F(\psi) = \psi|A$ for $\psi \in K(\overline{A})$ is a homomorphism and so $K(\overline{A}) \cong K(A)$. Hence, $\operatorname{ex} K(A) \cong \operatorname{ex} K(\overline{A})$ where $\operatorname{ex} K(A)$ is the set of extreme points of $K(A)$. Therefore $\operatorname{Ch}(A, X) = \operatorname{Ch}(\overline{A}, X)$. Since $\overline{\operatorname{Ch}}(A, X) = \Gamma(A, X)$ and $\overline{\operatorname{Ch}}(\overline{A}, X) = \Gamma(\overline{A}, X)$ we also have $\Gamma(A, X) = \Gamma(\overline{A}, X)$.

Theorem. *If A is a Banach function algebra on X then the restriction map $F \colon M_{\overline{A}} \to M_A$ is a homomorphism if and only if $\|\hat{f}\| = \|f\|_X$ for every $f \in A$.*

Remark. We usually write $M_{\overline{A}} \cong M_A$ whenever the above restriction map is a homeomorphism.

Proof. Let $F \colon M_{\overline{A}} \to M_A$ be the restriction map defined by $F(\psi) = \psi|_A$. Clearly $\phi = \psi|_A$ is a nonzero complex homomorphism on A and so $F(\psi) = \phi \in M_A$ for every $\psi \in M_{\overline{A}}$. If $f \in \overline{A}$ and $\phi \in M_A$ there exists a sequence $\{f_n\}$ in A such that $\|f_n - f\|_X \underset{n\to\infty}{\longrightarrow} 0$ and so by the hypothesis $|\phi(f_n) - \phi(f_m)| \leq \|f_n - f_m\|_X \underset{n,m\to\infty}{\longrightarrow} 0$. Hence, $\lim_{n\to\infty} \phi(f_n)$ exists and we can define ψ on $\overline{A}$ by $\psi(f) = \lim_{n\to\infty} \phi(f_n)$. It is easy to see that $\psi \in M_{\overline{A}}$ and in fact ψ is the extension of ϕ to $\overline{A}$, i.e. $F(\psi) = \phi$. Therefore F is one–one and onto. Considering the weak *-topology of M_A and $M_{\overline{A}}$, it is easy to show that F and F^{-1} are continuous. Hence F is a homomorphism and so $M_A \cong M_{\overline{A}}$.

Conversely let the restriction map $F \colon M_{\overline{A}} \to M_A$ be a homomorphism. If $\phi \in M_A$ and $f \in A$, then there exists $\psi \in M_{\overline{A}}$ such that $\phi = F(\psi) = \psi|A$ and so $|\phi(f)| = |\psi(f)| \leq \|f\|_X$. Hence for every $f \in A$, $\|\hat{f}\|_{M_A} = \sup_{\phi \in M_A} |\phi(f)| \leq \|f\|_X$.

Note that if A is a natural Banach function algebra on X, then $\overline{A}$ is also natural (uniform algebra) on X, but the converse is not true.

Corollary 1. *If A and B are Banach function algebras on X such that*

(i) $A \subseteq B$,

(ii) *there exists $K > 0$ such that for every $f \in A$, $\|f\|_A \leq K\|f\|_B$,*

(iii) *B is natural,*

then $M_A \cong M_{\overline{A}}$. In particular if $\overline{A} = \overline{B}$ then A is also natural.

Note that with the above hypotheses A is in fact a closed subalgebra of B under the norm of B.

Proof. By the spectral radius theorem and the above hypotheses for every $f \in A$.

$$\|\hat{f}\|_{M_A} = \lim_{n\to\infty} \|f^n\|_A^{1/n} \leq \lim_{n\to\infty} K^{1/n}\|f^n\|_B^{1/n} = \|\hat{f}\|_{M_B} = \|f\|_X .$$

Hence by the theorem $M_A \cong M_{\overline{A}}$. Since B is natural $\overline{B}$ is also natural. Thus if $\overline{A} = \overline{B}$ then $\overline{A}$ is also natural and so $M_A \cong X$, i.e. A is also natural.

The converse of Corollary 1 is not true, in the following sense: If A and B are Banach function algebras on X with the properties (i), (ii), $\overline{A} = \overline{B}$ and A is natural, then B may not be natural as the following example shows.

Example. Let

$$h(x) = \sum_{n=1}^{\infty} 2^{-n} \sin(2^{n^2} \pi x) .$$

Clearly h is continuous and $|h(x)| < 1$ on $\mathbf{R}$ and it is possible to prove that for every real x and positive integer n there exists $t \in [x, x + 2^{-n^2}]$ such that $|h(t) - h(x)| > 2^{-n-2}$.

Let $X = [0, 1]$, $A = D^1(X)$ and

$$B = \left\{ f : f = \sum_{n=0}^{\infty} f_n \cdot h^n ,\ f_n \in A ,\ \sum_{n=0}^{\infty} \|f_n\|_{D^1} < \infty \right\} .$$

Clearly $A \subseteq B$ and B is a function algebra on X. In order to define a norm on B we first prove the above representation of elements of B is unique.

Suppose $\sum_{n=0}^{\infty} f_n \cdot h^n = 0$ on X and $\sum_{n=0}^{\infty} \|f_n\|_{D^1} \leq 1$. To prove $f_n = 0$ on X for all $n \geq 0$, let $0 < x < 1$ and $0 < \varepsilon$ be small enough such that $\varepsilon + x < 1$. Since $f_n \in D^1(X)$ there exists $0 < \theta < 1$ such that $f_n(\varepsilon + x) - f_n(x) = \varepsilon f_n'(\theta)$. Therefore

$$\begin{aligned} \left| \sum_{n=0}^{\infty} f_n(x) h^n(\varepsilon + x) \right| &= \left| \sum_{n=0}^{\infty} (f_n(x) - f_n(\varepsilon + x)) h^n(\varepsilon + x) \right| \\ &\leq \sum_{n=0}^{\infty} \varepsilon \|f_n'\|_X < \varepsilon \sum_{n=0}^{\infty} \|f_n\|_{D^1} \leq \varepsilon . \end{aligned}$$

Now we define the sequence $\{a_n\}$ by

$$\sum_{n=0}^{\infty} f_n(x)[h(x)+t]^n = \sum_{n=0}^{\infty} a_n t^n$$

for small enough $|t|$. If the a_n are not all zero and $m = \min\{n \colon a_n \neq 0\}$, then for small enough ε, writing $h(\varepsilon + x) - h(x) = t$, we have

$$\left|\sum_{n=0}^{\infty} f_n(x)h^n(\varepsilon + x)\right| = \left|\sum_{n=m}^{\infty} a_n t^n\right| > \frac{1}{2}|a_m|\,|t|^m.$$

Therefore for small enough $\varepsilon > 0$, $\frac{1}{2}|a_m|\,|t|^m = \frac{1}{2}|a_m|\,|h(\varepsilon+x)-h(x)|^m < \varepsilon$, which contradicts the property of h. Hence, $a_n = 0$ and so $f_n(x) = 0$ for all $n \geq 0$. Since x is arbitrary, $f_n = 0$ on X for all $n \geq 0$.

Now we can define a norm on B by $\|f\|_B = \sum_{n=0}^{\infty} \|f_n\|_{D^1} (f \in B)$. It is easy to show that $(B, \|\cdot\|_B)$ is a Banach function algebra on X. Since A is self-adjoint by the Stone–Weierstrass theorem $\overline{A} = C(X)$ and so $\overline{A} = \overline{B} = C(X)$. Moreover for every $f \in A$, $\|f\|_A = \|f\|_B$. Although A and $\overline{B}$ are natural, B is not natural. Because for every $x \in X$ and every (non-real) complex number λ in the closed unit disk we can define an element $\phi_{x,\lambda}$ of M_B by

$$\phi_{x,\lambda}\left(\sum_{n=0}^{\infty} f_n \cdot h^n\right) = \sum_{n=0}^{\infty} f_n(x)\lambda^n.$$

It is easy to see that $\phi_{x,\lambda} \in M_B$ but it is not an evaluation homomorphism on X. Note that $h \in B$ but $h \notin D^1(X)$.

Now let X be a uniformly regular compact plane set and the sequence $\{M_k\}$ satisfy condition $(*)$. It is easily seen that $\overline{D}_P(X, \{M_k\}) = P(X)$. If $P_k \to \infty$ as $k \to \infty$ then every rational function with poles off X belongs to $D_R(X, \{M_k\})$ and so $\overline{D}_R(X, \{M_k\}) = \overline{D}(X, \{M_k\}) = R(X)$. Also it follows that:

(i) $\mathrm{Ch}(D_P, X) = \mathrm{Ch}(P(X), X) \subseteq \partial\widehat{X}$,
(ii) $\Gamma(D_P, X) = \Gamma(P(X), X) = \partial\widehat{X}$,
(iii) $\mathrm{Ch}(D_R, X) = \mathrm{Ch}(D, X) = \mathrm{Ch}(R(X), X) \subseteq \partial X$,
(iv) $\Gamma(D_R, X) = \Gamma(D, X) = \Gamma(R(X), X) = \partial X$,

where ∂X is the topological boundary of X.

As an application of Corollary 1, we give another proof of a theorem of Dales and Davie [1].

Corollary 2. *Let X be a uniformly regular compact plane set and the sequence $\{M_k\}$ satisfy condition $(*)$. If $P_k \to \infty$ as $k \to \infty$ then $M_{D_P} \cong \widehat{X}$.*

Proof. Clearly $D_P \subseteq D_R$ and for every $f \in D_P$, $\|f\|_{D_P} = \|f\|_{D_R}$. Since D_R is natural, by Corollary 1, $M_{\overline{D}_P} \cong M_{D_P}$. But $\overline{D}_P = P(X)$ and $M_{P(X)} \cong \widehat{X}$. So we have $M_{D_P} \cong \widehat{X}$.

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