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INVERSE-CLOSED CARLEMAN ALGEBRAS OF INFINITELY DIFFERENTIABLE FUNCTIONS

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ABSTRACT. We characterize the classes $\mathscr{E}_M(I)$ and $\mathscr{E}_M^*(I)$ of infinitely differentiable functions which are inverse-closed thereby giving a complete solution to a problem first posed by W. Rudin [11] and solved by him and J. Boman and L. Hörmander [2] for classes $\mathscr{E}_M(\mathbf{R})$ alone.

1. INTRODUCTION

Given a positive sequence $M = \{M_n\}$ and an interval I, let $\mathscr{E}_M(I)$ denote the Carleman class of all infinitely differentiable complex functions f defined on I for which $\sup_n\{\|f^{(n)}\|_\infty/M_n\}^{1/n} < \infty$. A class $\mathscr{E}_M(I)$ is said to be inverse-closed if f^{-1} is in $\mathscr{E}_M(I)$ whenever f is in $\mathscr{E}_M(I)$ and is bounded away from zero. More generally, analytic functions are said to operate on $\mathscr{E}_M(I)$ if for any f in $\mathscr{E}_M(I)$ and for any g analytic in an open set containing the closure of the range of f, $g \circ f$ is in $\mathscr{E}_M(I)$. A sequence M is called log-convex if $M_n^2 \le M_{n-1}M_{n+1}$ for $n \ge 1$.

In his paper on the symbolic calculus for the algebra of real functions which are Fourier transforms of functions in $L^1(G)$, P. Malliavin [8] (see J.-P. Kahane [7] for this and other related results) proved the following:

Theorem A. *If M is a log-convex sequence and the sequence $A \equiv \{A_n = (M_n/n!)^{1/n}\}$ is almost increasing in the sense that there exists a constant $K > 0$ such that $A_m \le KA_n$ for each m and n with $m < n$, then the class $\mathscr{E}_M(I)$ is inverse-closed.*

The problem as to whether the converse of Theorem A holds was first taken up by W. Rudin [11] who proved that it is so if $\mathscr{E}_M(R)$ is a non-quasi-analytic class of 2π-periodic functions, a restriction that was later removed by J. Boman and

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L Hörmander [2]. Thus even when the Carleman classes on $I \neq \mathbf{R}$ are defined by log-convex sequences, the converse of Theorem A is not known.

It is clear that Malliavin's sufficiency condition is applicable only to classes $\mathscr{C}_M(I)$ which are defined by log-convex sequences. However there exist classes $\mathscr{C}_M(I)$ which cannot be so defined. In fact, H. Cartan [5] has shown that if I is closed, then $\mathscr{C}_M(I) \not\equiv \mathscr{C}_{M^c}(I)$ even if $\mathscr{C}_{n!}(I) \subseteq \mathscr{C}_M(I)$, where M^c denotes the largest log-convex minorant of M. Thus for classes $\mathscr{C}_M(I)$ not defined by log-convex sequences, the problem of finding necessary and sufficient condition on M in order that they be inverse-closed over arbitrary I, has remained open so far (see, however, [12]). The same holds for local Carleman classes $\mathscr{C}_M^*(I)$ of functions which belong to class $\mathscr{C}_M(J)$ for each compact subinterval J of I.

In this paper, we give a complete solution of the problem for these classes. Instead of limiting a priori to classes defined by log-convex sequences M, we consider arbitrary classes which, as is well known (see [10]), can also be defined by regularized sequences M^r which vary according to the nature of the interval I. Using either the characteristic functions of these classes or Baire's category theorem applied to certain Fréchet spaces, we are able to show that they are inverse-closed if and only if $\{(M_n^r/n!)^{1/n}\}$ is almost increasing. The techniques employed here are different. They are simpler than those used for $I = \mathbf{R}$ in [11] and [2] and enable us to solve, in particular, the problem of characterization of inverse-closed algebras $\mathscr{C}_M^{2\pi}([-\pi, \pi])$ of 2π-periodic functions, posed by W. Rudin in [11].

The inverse-closed algebras $\mathscr{C}_M(I)$ and $\mathscr{C}_M^*(I)$ are, respectively, inductive limits of Banach and Fréchet spaces. Although, with the usual seminorms, they are not locally convex algebras (see [9] for relevant definitions), we can describe their (algebraic) maximal ideals and complex homomorphisms. Thus every maximal ideal of the inverse-closed algebra $\mathscr{C}_M(I)$ is of the form $\mathscr{I}_x = \{f \in \mathscr{C}_M(I) : f(x) = 0\}$ for some $x \in I$ and every complex homeomorphism is a point evaluation.

We may remark that J. Bruna [3] has proved that if M is log-convex, then the differentiable Beurling classes and their projective limits are inverse-closed if and only if $\{(M_n/n!)^{1/n}\}$ is almost increasing.

2. Inverse-closed Carleman classes

A Carleman class $X = \mathscr{C}_M(\mathbf{R})$ is always an algebra. In fact, if

$$\liminf_{n\to\infty} M_n^{1/n} = 0, \quad X \equiv \{\text{const}\}$$

and if

$$0 < \liminf_{n\to\infty} M_n^{1/n} < \infty, \quad X \equiv \mathscr{C}_1(\mathbf{R}).$$

In both these cases X is an algebra. Suppose now that $\lim_{n\to\infty} M_n^{1/n} = \infty$. If we set

$$T_M(r) = \sup_{n\geq 1} \frac{r^n}{M_n},$$

then

$$(1) \qquad M_n^c = \sup_{r\geq 1} \frac{r^n}{T_{sc}(r)}, \qquad T_{sc}(r) = \sup_{n\geq 1} \frac{r^n}{M_n^c} = T_{M^c}(r).$$

Since $X \equiv \mathscr{C}_{M^c}(\mathbf{R})$ (see [10], p. 226), and M^c is log-convex, using the Leibnitz formula for successive derivatives of a product, it is easily seen that X is an algebra.

Similarly $X = \mathscr{C}_M(\mathbf{R}^+)$ is always an algebra. In fact, if $\liminf_{n\to\infty} nM_n^{1/n} = 0$, $X \equiv \{\text{const}\}$ and if $0 < \liminf_{n\to\infty} nM_n^{1/n} < \infty$ then $X \equiv \mathscr{C}_{n^{-n}}(\mathbf{R}_+)$. Suppose now that $\lim_{n\to\infty} nM_n^{1/n} = \infty$. Then $X \equiv \mathscr{C}_{M^d}(\mathbf{R}_+)$ (see [10], p. 226), where $M^d \equiv \{M_n^d\}$ is defined by setting $n^n M_n^d = (n^n M_n)^c$ $(n \geq 1)$. If we put

$$T_M^*(r) = \sup_{n\geq 1} \frac{r^n}{n^n M_n} \quad (r \geq 1),$$

then

$$(2) \qquad n^n M_n^d = \sup_{r\geq 1} \frac{r^n}{T_{sc}^*(r)} \quad \text{and} \quad T_{sc}^*(r) = \sup_{n\geq 1} \frac{r^n}{n^n M_n^d} = T_{M^d}^*(r).$$

Here, as before, in all the above three cases $\mathscr{C}_M(\mathbf{R}_+)$ is an algebra.

A Carleman class $\mathscr{C}_M(I)$ consisting of the constants alone is always inverse-closed. For nontrivial inverse-closed Carleman classes, we have the following characterization.

Theorem 1. *Let $X = \mathscr{C}_M(\mathbf{R})$ or $\mathscr{C}_M(\mathbf{R}_+)$. If X is nontrivial, then the following assertions are equivalent*:

(a) $\lim_{n\to\infty} M_n^{1/n} = \infty$ *and the sequence A is almost increasing.*
(b) *Analytic functions operate on X.*
(c) *X is an inverse-closed algebra.*

Here $A = \{A_n\}$, where $A_n = (M_n^c/n!)^{1/n}$ or $(M_n^d/n!)^{1/n}$ $(n \geq 1)$ according as $X = \mathscr{C}_M(\mathbf{R})$ or $\mathscr{C}_M(\mathbf{R}_+)$.

Proof. (i) Let $X = \mathscr{C}_M(\mathbf{R})$. Suppose that (a) holds. Then $X \equiv \mathscr{C}_{M^c}(\mathbf{R})$. That analytic functions operate on X now follows since A is almost increasing, if we use Faà di Bruno's formula *viz.*,

$$(3) \qquad (g\circ f)^{(n)}(x) = \sum \frac{n!}{k_1!\cdots k_n!} g^{(k)}(f(x)) \left(\frac{f'(x)}{1!}\right)^{k_1} \cdots \left(\frac{f^{(n)}(x)}{n!}\right)^{k_n},$$

where

$$2^{n-1} = \sum \frac{k!}{k_1!\cdots k_n!},$$

and the summation in the two cases is over all n-tuples $(k_1, \dots, k_n)$ such that $k_1 + \cdots + k_n = k$ and $k_1 + 2k_2 + \cdots + nk_n = n$ $(0 \leq k \leq n)$. Thus (b) holds.

Trivially (b) implies (c). It remains to be shown that (c) implies (a). Let (c) hold. Since X is nontrivial, we cannot have $\liminf_{n\to\infty} M_n^{1/n} = 0$, for otherwise, we would have $\mathscr{C}_M(\mathbf{R}) \equiv \{\text{const}\}$. We cannot have $0 < \liminf_{n\to\infty} M_n^{1/n} < \infty$ either, for then we would have $\mathscr{C}_M(\mathbf{R}) \equiv \mathscr{C}_1(\mathbf{R})$ which is not inverse-closed contradicting the hypothesis, since f, where $f(x) = 2 + \sin x$ is in $\mathscr{C}_1(\mathbf{R})$ but not its inverse (see [1], p. 25). Thus $\lim_{n\to\infty} M_n^{1/n} = \infty$. It follows from (1) that there exists a positive sequence $\{r_n\}$ such that $r_n^n = M_n^c T_{M^c}(r_n)$ $(n \geq 1)$. The function

$$f(x) = \sum_{j=1}^{\infty} \frac{e^{ir_j x}}{2^j T_{M^c}(r_j)}, \tag{4}$$

is a characteristic function of X since it is easily seen that

1°. $|f^{(n)}(x)| \leq M_n^c$ $(n \geq 0;\, x \in \mathbf{R})$,

2°. $f^{(n)}(0) = i^n s_n$ $(n \geq 0)$, where $s_n \geq 2^{-n} M_n^c$.

If $\lambda > 1 + \|f\|_\infty$, then $\lambda - f \in X$ and since it does not vanish on $\mathbf{R}$ and X is inverse-closed, $(\lambda - f)^{-1}$ belongs to X. From (3), we get

$$\sum \frac{k!}{k_1! \cdots k_n!} (\lambda - f(0))^{-k-1} \left(\frac{M_1^c}{1!2}\right)^{k_1} \cdots \left(\frac{M_n^c}{n!2^n}\right)^{k_n} \leq AB^n \frac{M_n^c}{n!}.$$

Let $p > 1$ be a fixed integer and suppose that $n = pk$. The term in the above inequality that corresponds to the choice $k_p = k$ and $k_q = 0$ for $q \neq p$ does not exceed that on the right so that

$$\left(\frac{M_p^c}{p!}\right)^{1/p} \leq K_1 \left(\frac{M_n^c}{n!}\right)^{1/n}.$$

If n is not a multiple of p, let $pm \leq n \leq p(m+1)$. Since $\{(M_n^c)^{1/n}\}$ is increasing, we get

$$\left(\frac{M_n^c}{n!}\right)^{1/n} \geq \left(\frac{M_{pm}^c}{(pm)!}\right)^{1/pm} \frac{((pm)!)^{1/pm}}{(n!)^{1/n}} \geq \frac{1}{K_2} \left(\frac{M_p^c}{p!}\right)^{1/p}$$

so that (a) holds.

(ii) Let $X = \mathscr{C}_M(\mathbf{R}_+)$. Suppose that (a) holds. Since $\lim_{n\to\infty} nM_n^{1/n} = \infty$, $X \equiv \mathscr{C}_{M^d}(\mathbf{R}_+)$. As in (i), (a) implies (b) and (b) implies (c). Suppose that (c) holds. Since X is nontrivial, we cannot have $\liminf_{n\to\infty} nM_n^{1/n} = 0$. We cannot have $0 < \liminf_{n\to\infty} nM_n^{1/n} < \infty$. For then $X \equiv \mathscr{C}_{n^{-n}}(\mathbf{R}_+)$ which is not inverse-closed. In fact, let

$$h(x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{(2k+3)!}, \qquad (x \in \mathbf{R}_+).$$

Using the properties of Mittag–Leffler function, we prove that

$$|h^{(n)}(x)| \leq 2e^n n^{-n} \qquad (n \geq 1), \quad (x \in \mathbf{R}_+), \tag{5}$$

so that $h \in X$. Choosing $\lambda > 1 + \|h\|_\infty$ and applying (3), we get

$$[(\lambda - h(x))^{-1}]^{(2m)}_{x=0} \geq \frac{(2m)!}{(7!)^m m!(\lambda - h(0))^{m+1}}.$$

It follows that

$$\limsup_{n\to\infty} n \left(\max_{x\in\mathbf{R}_+} |\{(\lambda - h(x))^{-1}\}^{(n)}| \right)^{1/n} = \infty.$$

Hence $(\lambda - h)^{-1} \notin X$, i.e. X is not inverse-closed contradicting the hypothesis. Thus $\lim_{n\to\infty} nM_n^{1/n} = \infty$ and $X \equiv \mathscr{C}_{M^d}(\mathbf{R}_+)$. It follows from (2) that there exists a positive sequence $\{r_n\}$ such that $r_n^n = n^n M_n^d T^*_{M^d}(r_n)$ $(n \geq 1)$. The function

$$f(x) = \sum_{j=1}^{\infty} \frac{h(r_j x)}{2^j T^*_{M^d}(r_j)} \tag{6}$$

is a characteristic function of X since, by (5),

1°. $|f^{(n)}(x)| \leq 2e^n M_n^d$ $(n \geq 0;\, x \in \mathbf{R}_+)$,

2°. $f^{(n)}(0) = (-1)^n s_n$, where $s_n \geq \mu^n M_n^d$ $(\mu > 0)$.

If we choose $\lambda > 1 + \|f\|_\infty$, then reasoning as in (i), we conclude that if $p > 1$ is a fixed integer and n is a multiple of p, say, $n = pk$, we get

$$\frac{1}{(\lambda - f(0))^{k+1}} \left(\frac{p^p p! M_p^d}{(2p+3)! 2^p} \right)^k \leq AB^n \frac{M_n^d}{n!}$$

and consequently

$$\left(\frac{M_p^d}{p!} \right)^{1/p} \leq K \left(\frac{M_n^d}{n!} \right)^{1/n}.$$

The result for arbitrary n now follows as in (i) if we remember that $\{(n^n M_n^d)^{\frac{1}{n}}\}$ is increasing.

3. Inverse-closed local Carleman classes

We now proceed to characterize the inverse-closed local Carleman classes.

If I is a finite or infinite open interval, then $\mathscr{C}^*_M(I) \equiv \mathscr{C}^*_{M^0}(I)$ (see [10], p. 223), where $M^0 = \{M_n^0\}$ is defined by setting

$$S_M(r) = \max_{n\leq r} \frac{r^n}{M_n}, \qquad M_n^0 = \sup_{r\geq n} \frac{r^n}{S_M(r)}.$$

Then $S_M(r) = S_{M^0}(r)$. But if I is an arbitrary interval then $\mathscr{C}^*_M(I) \equiv \mathscr{C}^*_{M^f}(I)$ (see [10], p. 223 and [4], p. 718), where $M^f = \{M_n^f\}$ is defined by setting

$$U_{sc}(r) = \max_{n\leq r} \frac{r^{2n}}{n^n M_n} \quad \text{and} \quad n^n M_n^f = \sup_{r\geq n} \frac{r^{2n}}{U_M(r)}.$$

Then $U_M(r) = U_{M^f}(r)$.

A local Carleman class $X = \mathscr{C}_M^*(I)$ is not always an algebra (see [6], p. 337). However, it is so if it contains the local analytic class. In fact, if I is open, then, as shown by H. Cartan (see [5], p. 7), $X \equiv \mathscr{C}_{M^c}^*(I)$ so that $X \equiv \mathscr{C}_{M^0}^*(I) \equiv \mathscr{C}_{M^c}^*(I)$ is an algebra. The same is true for local Carleman class $\mathscr{C}_M^*(I)$ defined on a closed or a semi-closed interval I. However in this case the following analogue of Cartan's result holds.

Lemma I. *If $X = \mathscr{C}_M^*(I) \supseteq \mathscr{C}_{n!}^*(I)$, then $X \equiv \mathscr{C}_{M^f}^*(I) \equiv \mathscr{C}_{M^d}^*(I)$ for any interval I. Consequently X is an algebra.*

Proof. Since $X \equiv \mathscr{C}_M^*(I) \supseteq \mathscr{C}_{n!}^*(I)$, it follows that $M_n \geq k^n M_0 n!$ $(n \geq 0)$. Choose $f \in \mathscr{C}_M^*(I)$. Let J be a compact subinterval of I and let I' be a compact subinterval of I such that each point x of J is in a subinterval of fixed length λ (< 1) contained in I', where $|f^{(n)}(x)| \leq K\sigma^n M_n$ $(n \geq 0)$.

Choose σ so large that $\lambda\sigma \geq k^{-1}$. Clearly $n!k \;\; M_0 \leq K(\lambda\sigma)^n M_n$ $(n \geq 0)$. Let $\{n_i\}$ be the sequence of principal indices for $M' = \{K\sigma^n M_n\}$ and let $n_i < n < n_{i+1}$. Since $(K\sigma^n n^n M_n^d)$ is log-convex, applying Cartan-Gorny inequalities (see [10], p. 219), we conclude that for any x in J

$$\begin{aligned} |f^{(n)}(x)| &\leq 2(e^2 rp^{-1})^p (K(\lambda\sigma)^{n_i} M_{n_i})^{q/r} (K(\lambda\sigma)^{n_{i+1}} M_{n_{i+1}})^{p/r} \\ &\leq 2K(e^2 rp^{-1})^p n_i^{-n_i q/r} n_{i+1}^{-n_{i+1} p/r} (\lambda\sigma n)^n M_n^d , \end{aligned}$$

where $p = n - n_i$, $q = n_{i+1} - n$, $r = n_{i+1} - n_i$. Set $u = n_{i+1}/n_i$. Then

$$n_i^{-n_i q/r} n_{i+1}^{-n_{i+1} p/r} = (u^{u/(u-1)}(u-1))^{n-n_i} \leq 1 .$$

Hence

$$|f^{(n)}(x)| \leq \mu^n M_n^d \qquad (x \in J).$$

Since $M_{n_i} = M_{n_i}^d$, this inequality holds for all $n \geq 0$. Thus $f \in \mathscr{C}_{M^d}^*(I)$. Clearly $\mathscr{C}_{M^d}^*(I)$ is an algebra.

Thus a local Carleman algebra $X = \mathscr{C}_M^*(I) \supseteq \mathscr{C}_{n!}^*(I)$ has two regularizations *viz.* $X \equiv \mathscr{C}_{M^0}^*(I) \equiv \mathscr{C}_{M^c}^*(I)$ or $X \equiv \mathscr{C}_{M^f}^*(I) \equiv \mathscr{C}_{M^d}^*(I)$ according as I is open or arbitrary.

Although, in general, it is not true, the equivalence of classes in these two cases does imply that the sequences M^0 and M^c and the sequences M^f and M^d are equivalent in the sense that for some constants $\alpha > 0$ and $\beta > 0$

$$(7) \qquad \text{(a)} \;\; \beta^n M_n^0 \leq M_n^c \leq \alpha^n M_n^0 , \qquad \text{(b)} \;\; \beta^n M_n^f \leq M_n^d \leq \alpha^n M_n^f \quad (n \geq 1) .$$

The second halves of (7)(a) and (b) are obviously true with $\alpha = 1$. For I finite, (7) follows from the inclusion theorems of H. Cartan and S. Mandelbrojt (see [10], p. 238). The arguments used by these authors fail when I is infinite. However, using Baire's category theorem, we get the same result valid in all cases.

Lemma II. *For any finite or infinite open interval I, $\mathscr{C}_M^*(I) \subseteq \mathscr{C}_N^*(I)$ if and only if $(M_n^0)^{1/n} = O[(N_n^0)^{1/n}]$ or $(M_n^0)^{1/n} = O[(N_n)^{1/n}]$.*

Proof. Since $\mathscr{C}_M^*(I) \equiv \mathscr{C}_{M^0}^*(I)$ and $\mathscr{C}_N^*(I) \equiv \mathscr{C}_{N^0}^*(I)$, the sufficiency of the conditions is obvious. So we prove that they are necessary. We may suppose, without loss of generality, that $I =]-1, 1[$ or $]-1, \infty[$ or $\mathbf{R}$. For each n, choose an integer $h_n \geq n$ such that

$$\frac{h_n^n}{S_{M^0}(r)} \geq \frac{M_n^0}{e}. \tag{8}$$

For $n \geq 2$, set $I_n = [-1 + (1/h_n), 1 - (1/h_n)]$ or $[-1 + (1/h_n), h_n]$ or $[-h_n, h_n]$ according as $I =]-1, 1[$ or $]-1, \infty[$ or $[-\infty, \infty[$. Let $\mathscr{F}$ denote the class of functions $f \in \mathscr{C}^\infty(I)$ such that

$$p_k(f) = \sup_{n \geq 0} \max_{x \in I_k} \frac{|f^{(n)}(x)|}{M_n^0}, \qquad k = 2, 3, \ldots.$$

$\mathscr{F}$ is a Fréchet space with seminorms $\{p_k\}$ and $\mathscr{F} \subseteq \mathscr{C}_{M^0}^*(I)$. Set

$$V_j = \{f \in \mathscr{F} : |f^{(n)}(0)| \leq j^{n+1} N_n^0 \ (n \geq 0\}, \qquad j = 1, 2, \ldots.$$

If $f \in \mathscr{F}$, $f \in \mathscr{C}_{M^0}^*(I) \subseteq \mathscr{C}_{N^0}^*(I)$. Hence $f \in V_j$ for some j. Thus $\mathscr{F} = \bigcup_1^\infty V_j$. Clearly, V_j's are closed in $\mathscr{F}$. Hence, by Baire's category theorem, there exists a seminorm p_r, a $\delta > 0$ and a V_s such that $p_r(f) \leq \delta$ implies that $f \in V_s$. Set $\alpha = 1/6h_r$ and let

$$f(x) = \frac{\delta Z_{h_n}(\alpha x)}{2S_{M^0}(h_n)},$$

where

$$Z_n(x) = (-1)^{[n/2]} T_n(x) + (-1)^{[(n-1)/2]} T_{n-1}(x),$$

$T_n(x)$ denoting the Chebyshev polynomial of degree n and $[t]$, the integral part of t. For $x \in I_q$, $f^{(k)}(x) \equiv 0$ if $k > h_n$, and

$$|f^{(k)}(x)| = \frac{1}{2}\delta\alpha^k \frac{|Z_{h_n}^{(k)}(\alpha x)|}{S_{M^0}(h_n)} \leq A_q M_k^0$$

if $1 \leq k \leq h_n$, since

$$|T_n^{(k)}(x)| \leq (4n)^k e^{4ne|x|}, \qquad x \in \mathbf{R}. \tag{9}$$

Thus $f \in \mathscr{F}$. If $x \in I_r$, then

$$|f^{(k)}(x)| \leq \delta \frac{h_n^k}{S_{M^0}(h_n)} \frac{3^k \alpha^k}{(1 - \alpha^2 h_r^2)^k} \leq \delta M_k^0$$

since (see [10], p. 206) for $-1 < x < 1$:

$$|Z_n^{(k)}(x)| \leq \frac{2.3^k n^k}{(1 - x^2)^k} \qquad (0 \leq k \leq n).$$

Hence $p_r(f) \le \delta$. Then $f \in V_s$ and so for every $k \ge 1$,

$$\frac{1}{2}\delta\alpha^k \frac{|Z_{h_n}^{(k)}(0)|}{S_{M^0}(h_n)} \le s^{k+1} N_k^0 .$$

Since

$$(n/e)^k \le |Z_n^{(k)}(0)| \le n^k ,$$

it follows that

$$\frac{1}{2}\delta \frac{\alpha^k}{e^k} \frac{h_n^k}{S_{M^0}(h_n)} \le s^{k+1} N_k^0 .$$

Choosing $k = n$ and using (8), we see that, for some $\beta > 0$,

$$\beta^n M_n^0 \le N_n^0 \qquad (n \ge 1) .$$

Thus the first condition is necessary and so also the second.

If we apply this lemma with $N_n = M_n^0$, we get the first half of (7)(a). To prove (7)(b), we need the following:

Lemma III. *If I is not an open interval, then $\mathscr{C}_M^*(I) \subseteq \mathscr{C}_N^*(I)$ if and only if $(M_n^f)^{1/n} = O[(N_n^f)^{1/n}]$ or $(M_n^f)^{1/n} = O[(N_n)^{1/n}]$.*

Proof. Since $\mathscr{C}_M^*(I) \equiv \mathscr{C}_{M^f}^*(I)$ and $\mathscr{C}_N^*(I) \equiv \mathscr{C}_{N^f}^*(I)$, we need prove only the necessity part of the lemma. Suppose $I = \mathbf{R}_+$. Let $I_h = [0, h]$ $(h \ge 2)$ and let $\mathscr{F}$ denote the class of functions $f \in \mathscr{C}^\infty(I)$ such that for each $h \ge 2$,

$$p_h(f) = \sup_{n \ge 0} \left(\max_{x \in I_h} |f^{(n)}(x)|/M_n^f \right) < \infty .$$

$\{p_h\}$ is a family of seminorms on $\mathscr{F}$ making $\mathscr{F}$ a Fréchet space. Let

$$V_j = \{ f \in \mathscr{F} : |f^{(n)}(1)| \le j^{n+1} N_n^f \quad (n \ge 1), \qquad j = 1, 2, \dots . \}$$

Clearly $\mathscr{F} = \bigcup_1^\infty V_j$ and V_j's are closed. Applying Baire's category theorem, we get a seminorm p_r, a $\delta > 0$ and a V_s such that $p_r(f) \le \delta$ implies that $f \in V_s$. Let

$$f(x) = \frac{\delta T_{k_n}(x/r)}{U_{M^f}(k_n)} , \tag{10}$$

where T_n denotes the Chebyshev polynomial of degree n and $\{k_n\}$ a sequence of integers chosen such that $k_n \ge n$ and

$$\frac{k_n^{2n}}{U_{M^f}(k_n)} \ge \frac{n^n M_n^f}{e^2} . \tag{11}$$

$f \in \mathscr{F}$ since $f^{(m)}(x) = 0$ for $x \in I_h$ and $m > k_n$ and, by (9),

$$|f^{(m)}(x)| \le A_h M_n^f , \qquad x \in I_h , \quad m \le k_n .$$

Since, for $x \in [-1, 1]$, $|T_n^{(j)}(x)| \leq (en^2/2j)^j$, we have

$$p_r(f) \leq \frac{\delta}{U_{M^f}(k_n)} \sup_{m \leq k_n} \frac{k_n^{2m}}{m^m M_m^f} \leq \delta,$$

Therefore $f \in V_s$ and so for each $m \geq 1$.

$$\delta r^{-m} \left(\frac{2}{em}\right)^m \frac{k_n^{2m}}{U_{M^f}(k_n)} \leq s^{m+1} N_m.$$

Since $T_n^{(j)}(1) \geq (2n^2/ej)^j$. Choosing $m = n$, we get from (11)

$$\delta r^{-n} \left(\frac{2}{en}\right)^n e^{-2} n^n M_n^f \leq s^{n+1} N_n^f.$$

Thus $(M_n^f)^{1/n} = O[(N_n^f)^{1/n}]$ and so also $(M_n^f)^{1/n} = O[(N_n)^{1/n}]$.

If I is finite, we may take it to be $[-1, 1]$ or $]-1, 1]$. Let $\mathscr{B}$ denote the Banach space of functions $f \in \mathscr{C}^\infty(I)$ such that

$$\|f\|_{\mathscr{B}} = \sup_{n \geq 0} \left(\max_{0 \leq x \leq 1} |f^{(n)}(x)|/M_n^f \right) > \infty.$$

Clearly $\mathscr{B}$ is a union of the closed sets V_j, where

$$V_j = \{f \in \mathscr{B} : |f^{(n)}(1)| \leq j^{n+1} N_n^f \quad (n \geq 1), \qquad j = 1, 2, \ldots.\}$$

Arguing as before with $r = (e/2)$ in (10), we complete the proof.

If $X = \mathscr{C}_M^*(I) \supseteq \mathscr{C}_{n!}^*(I)$, we choose $N_n = M_n^d$ in Lemma III, and get the first half of (7)(b).

The following theorem characterizes the inverse-closed local Carleman classes:

Theorem 2. *Let $X = \mathscr{C}_M^*(I)$. The following assertions are equivalent:*

(a) *The sequence $A = \{A_n\}$ is almost increasing.*
(b) *Analytic functions operate on X.*
(c) *X is an inverse-closed algebra.*

Here $A_n = (M_n^0/n!)^{1/n}$ or $(M_n^f/n!)^{1/n}$ according as I is open or not.

Proof. (i) Let $X \equiv \mathscr{C}_M^*(I)$, where I is an open interval which we may suppose, without loss of generality, to be $]-1, 1[$ or $]-1, \infty[$ or $\mathbf{R}$. Let (a) hold. Since A is almost increasing, from (2), we conclude that analytic functions operate on X. This trivially implies that X is an algebra. Thus (b) holds.

Suppose that (c) holds. If we choose $f(x) = 1+x^2$, then f and consequently f^{-1} belongs to X so that $n! \leq AB^n M_n$ $(n \geq 1)$. Thus $\mathscr{C}_{n!}^*(I) \subseteq X$ and $X \equiv \mathscr{C}_{M^c}^*(I)$. Moreover, reasoning as in (i) of the proof of Theorem 1 and using the function f constructed there, we see that $\{(M_n^c/n!)^{1/n}\}$ is almost increasing. But then, by (7), A is also almost increasing. Thus (c) holds.

(ii) Let $X \equiv \mathscr{C}_M^*(I)$, where I is not open. Then $X \equiv \mathscr{C}_{M^f}^*(I)$. Here we take $A_n = (M_n^f/n!)^{1/n}$ $(n \geq 1)$. As in (i), (a) implies (b) and (b) implies (c). We only need show that (c) implies (a).

Without loss of generality, we may suppose that $I = [0, 1]$ or $]0, 1]$ or $\mathbf{R}_+$. Since X is inverse-closed and $f \in X$, where $f(x) = 1 + x$, it follows that $f^{-1} \in X$ so that $X \supseteq \mathscr{C}_{n!}^*(I)$. But then, by Lemma I, $X \equiv \mathscr{C}_{M^d}^*(I)$. Hence the function f constructed in (ii) of the proof of Theorem 1 or its restriction is in X and so also its inverse f^{-1}. From this point on the same proof with obvious modifications goes through and we conclude that $\{(M_n^d/n!)^{1/n}\}$ is almost increasing. But then, by (7), A is almost increasing.

4. Remarks

We make a few concluding remarks.

1°. The following theorem which completes Theorem A characterizes inverse-closed Carleman and local Carleman algebras defined by log-convex sequences M.

Theorem 3. *Let $X = \mathscr{C}_M(I)$ or $\mathscr{C}_M^*(I)$, where M is log-convex. The following assertions are equivalent.*

(a) *$\{(M_n/n!)^{1/n}\}$ is almost increasing.*
(b) *Analytic functions operate on X.*
(c) *X is an inverse-closed algebra.*

Proof. It suffices to note that (c) implies (a) as in (i) of the proof of Theorem 1 since, by trivial modifications, the characteristic function constructed there becomes such a function for any class $\mathscr{C}_M(I)$ or $\mathscr{C}_M^*(I)$ with the desired properties.

2°. Theorems 1 and 2 characterize all inverse-closed Carleman and local Carleman algebras if we note that $\mathscr{C}_M(I) \equiv \mathscr{C}_M(\bar{I})$ and that for finite I, $\mathscr{C}_M(I) \equiv \mathscr{C}_M(\bar{I}) \equiv \mathscr{C}_M^*(I)$.

3°. If we repeat the proof of Theorem 1 for the class $X \equiv \mathscr{C}_M^{2\pi}([0, 2\pi])$, replacing the function f used there by the function

$$f(x) = \sum_{j=1}^{\infty} \frac{e^{i[r_j]x}}{2^j T_{M^c}(r_j)},$$

where $[t]$ denotes, as usual, the integral part of t, we conclude in response to the question raised by W. Rudin [11] that X is inverse-closed if and only if $\{(M_n^c/n!)^{1/n}\}$ is almost increasing.

REFERENCES

1. T. Bang, *Om quasianalytiske funktioner*, Nyt Nordlisk Forlag, Copenhagen, 1946.
2. J. Boman and L. Hörmander, *Classes of infinitely differentiable functions*, (mimeographed notes) (Swedish), Stockholm, 1962.
3. J. Bruna, *On inverse-closed algebras of infinitely differentiable functions*, Studia Math. **69** (1980), 61–67.
4. H. Cartan, *Solution d'une problème de Carleman pour un intervalle fermé fini*, C. R. Acad. Sci. Paris **208** (1939), 716–718.
5. ——, *Sur les classes de fonctions définies par des inégalités portant sur les dérivées successives*, Herman, Paris, 1940.
6. A. Gorny, *Contributions à l'études des fonctions dérivables d'une variable réele*, Acta Math. **71** (1939), 317–358.
7. J.-P. Kahane, *Séries de Fourier absolument convergentes*, Springer-Verlag, Berlin-Heidelberg-New York, 1970.
8. P. Malliavin, *Calcul symbolique et sous-algèbres de* $L^1(G)$, Bull. Soc. Math. France **87** (1959), 181–186.
9. A. Mallios, *Topological algebras. Selected topics*, North-Holland Math. Studies, **124**, Amsterdam, 1986.
10. S. Mandelbrojt, *Séries adhérentes, régularisatiosn des suites, applications*, Gauthier-Villars, Paris, 1952.
11. W. Rudin, *Division in algebras of infinitely differentiable functions*, J. Math. Mech. **11** (1962), 797–809.
12. J. A. Siddiqi, *Sur les classe de fonctions différentiables fermées par passage à l'inverse*, C. R. Acad. Sci. Paris **305** (1987), 9–12.

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