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COMMUTATORS AND Π-SUBGROUPS

ROLF BRANDL

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Dedicated to Professor B. H. Neumann on the occasion of his 80th birthday

ABSTRACT. The following statement is shown to be true for a large class of groups G: if every commutator in G is a Π-element, then G' is a Π-group

1. INTRODUCTION

It is well known that not every element contained in the commutator subgroup G' of a group G necessarily is a commutator. The smallest example is a group of order 96 (see [2, 3]. In fact, the nonabelian free groups show that, in general, there is no bound for the number of commutators needed to express an element of G' (see [7, p. 55]). This situation can occur even when G' is cyclic [8]. These examples show that G' can be much 'larger' than the set of all commutators. Indeed, the situation as described above seems to be the rule rather the exception. Although a long standing question of O. Ore [11] asks to show that every element in a finite simple group G is a commutator (using [4] this is easy once the character table of G is known), Marty Isaacs [6] has constructed perfect groups for which this is not true.

A positive result in this context could be something like: a 'good' property of the commutators prevails for the commutator subgroup. Apart from the trivial remark that if every commutator is the identity, then G is abelian, there seems to exist only the (highly nontrivial) result of I. D. Macdonald [9] saying that if all commutators in G are of order ≤ 2, then G' is of exponent 4. The groups constructed by B. H. Neumann [10] show that $\exp(G') = 4$ indeed can happen.

Macdonald comments in [9, p. 279] that "no one seems to have decided whether a group in which all commutators have finite, or bounded, order can have a nonperiodic commutator subgroup." A partial answer to this is the following main result of this paper:

Theorem. *Let G be hyper-(torsionfree or locally finite) and let Π be a set of primes. If every commutator in G is a Π-element, then G' is a Π-group. In particular, if every commutator has finite order, then G' is a torsion group.*

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We shall use standard notation throughout, see [5] or [12]. Moreover, a group G is called hyper-(torsionfree or locally finite) if every nonidentity factor group of G either has a nontrivial torsionfree or a nontrivial locally finite normal subgroup. This class of groups contains all finite groups and all soluble groups, for example. A group is called Π-closed if the set of all its Π-elements forms a subgroup.

2. The proofs

For reasons of clarity, we first prove some special cases of the theorem. A weaker version of Proposition 1 can be found in [1].

Proposition 1. *Let G be a finite group, let Π be a set of primes and let r be a positive integer. If $[x, {}_ry]$ is a Π-element for all $x, y \in G$, then $\gamma_\infty(G)$ is a Π-group. If $r = 1$, then G' is a Π-group.*

Proof. The hypothesis on G clearly is inherited by subgroups and quotients. Let p be a prime *not* contained in Π. We first show that G is p-nilpotent. Let G be a counterexample of least possible order. A result of Itô [5, p. 434] implies that G is a split extension of a normal p-subgroup P by a p'-group Q. By hypothesis, $z = [x, {}_ry]$ is a p'-element for all $x \in P$, $y \in Q$. As $z \in P$, we have $z = 1$. By [5, p. 350], this implies $[x, y] = 1$ and so $G = P \times Q$ is p-nilpotent, a contradiction.

Thus G is p-nilpotent for all primes p not contained in Π. So G is Π-closed and $G/O_\Pi(G)$ is nilpotent. This proves the first assertion. If $r = 1$, then every commutator in G belongs to $O_\Pi(G)$ and so G' is a Π-group as claimed. □

The following shows that in Proposition 1 there does not exist a positive integer k such that $\gamma_k(G)$ is a Π-group for every G.

Example. Let Π be a set of odd primes and let G be the wreath product of a group of order 2 by an infinite elementary abelian 2-group. Then $[x, {}_3y] = 1$ is a Π-element for all $x, y \in G$, but $\gamma_k(G)$ is a nontrivial 2-group for every k.

Proposition 2. *The theorem is true if $G/Z(G)$ is locally finite.*

Proof. Obviously, we can assume that G is finitely generated and so $Z = Z(G)$ has finite index in G. Thus Z is finitely generated and hence has a torsionfree subgroup S of finite index. Also, by a result of Schur (see [12, Theorem 4.12]), G' is finite. Thus G' and S have trivial intersection and hence $(G/S)'$ is isomorphic to G'. The result now follows from Proposition 1. □

Proof of the theorem. We can assume that G has no normal Π-subgroups. Let S be a normal subgroup of G maximal with respect to containing no Π-elements. This exists by Zorn's Lemma. By hypothesis, S is contained in the centre of

G. If $S = G$, then G is abelian and we are done. Otherwise, there exists a normal subgroup R of G properly containing S with R/S either torsionfree or locally finite. The maximality of S yields that R/S cannot be torsionfree, so R/S is locally finite. Here, Proposition 2 implies that R' is a Π-subgroup. Thus R' is a normal Π-subgroup of G, and hence it must be trivial. Thus R is abelian. The Π-elements of R form a normal subgroup of G, and so R does not contain any Π-elements. But this contradicts the choice of S. □

We are indebted to the referee for pointing out the following consequence of our result.

Corollary. *Let G be a group and assume that one of the following conditions hold.* (a) *G is linear, or* (b) *G is residually finite and G' is a torsion group.*
Then the theorem holds for G.

Proof. (a) If G is linear, then by Tits' alternative [13], either G contains a nonabelian free subgroup or G is soluble-by-finite. In the first case there are commutators of infinite order while in the second case the theorem applies.

(b) If G' is not a Π-group, then there exists an element $g \in G'$ of prime order p not in Π. Now let N be a normal subgroup of finite index in G that does not contain g. Then $gN \in (G/N)'$ and so $(G/N)'$ is not a Π-group, against the theorem. □

We add some obvious questions:

Remark. (a) By Macdonald's result [9], if $[x, y]^2 = 1$ for all $x, y \in G$, then G is centre-by-metabelian. It is not known whether $[x, y]^3 = 1$ for all $x, y \in G$ implies that G is soluble.

(b) It is not known whether there is a function f on the positive integers with the following property. Let G be a (finite) group and assume that $[x, y]^n = 1$ for all $x, y \in G$. Then $z^{f(n)} = 1$ for all $z \in G'$. Trivially, $f(1) = 1$ and Macdonald's result shows that one can take $f(2) = 4$. Does $f(3)$ exist?

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MATHEMATISCHES INSTITUT, AM HUBLAND 12, D-8700 WÜRZBURG, FEDERAL REPUBLIC OF GERMANY