AN ASYMPTOTIC BOUND
FOR THE ITERATES OF CERTAIN REAL FUNCTIONS
NEAR A CONTRACTIVE FIXED POINT

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Abstract. If \( x = 0 \) is a contractive fixed point for the function \( F \), then under certain conditions, the iterates \( F_k(a) \) are asymptotically equal to the numbers \( \xi_k \) defined by \( k = \int_{\xi_k}^{a} \frac{du}{G(u)} \). Using somewhat different hypotheses, we give a more precise bound on \( \frac{F_k(a)}{\xi_k} \).

Let \( F \) be a real function of the real variable \( x \), defined and continuous for \( 0 < x < A \), with \( 0 < F(x) < x \) for \( 0 < x < A \). The \( k \)th iterate of \( F \) will be denoted by \( F_k(F_0(x) \equiv x) \). Set the error \( x - F(x) = G(x) \). In the classical case where \( G(x) \sim Cx^\alpha, C > 0, \alpha > 1 \), the asymptotic formula \( F_k(a) \sim ((a - 1)Ck)^{1/1-\alpha} \) has been known for a long time (see [1] or [3]).

In the elementary note [2], the author and V. Drobot dealt with certain other cases and proved that when \( F \) converges to 0 "slowly" and certain other conditions are met, then the first asymptotic term is given by \( \xi_k \) where \( \xi_k \) is defined by the equation

\[
(*) \quad k = \int_{\xi_k}^{a} \frac{du}{G(u)}.
\]

In addition, we gave several heuristic arguments for \((*)\). The theorem proved in [2] follows.

**Theorem 0.** With the notation given previously, assume \( G \in C^1 \), \( G(0) = G'(0) = 0 \), and \( G \) is strictly increasing. Define

\[
H(x) = \int_{x}^{a} \frac{du}{G(u)}
\]

so that \( H(\xi_k) = k \). Assume that for each \( \mu > 1 \),

\[
\limsup_{x \to 0} \frac{H(\mu x)}{H(x)} < 1.
\]

Then \( \xi_k \sim F_k(a) \).

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The main purpose of this note is to give a bound for $F_k(a)/\xi_k$. We do this by imposing a new criterion for "slow convergence to 0." Moreover, under this condition, we can dispense with the unwanted assumption that $G'(0) = 0$. In fact, the result will apply to certain functions composed of line segments whose slopes take successively the values 0 and 1. Although, in the generality of our setting, we cannot get a true second asymptotic term, our error has the correct order of magnitude in the classical case.

**Theorem 1.** With the notation from above, assume $F$ and $G$ are increasing for $0 \leq x \leq A$. Let $K$ denote the function inverse to $H$, and set $Q = -K'$. Assume moreover that

$$\lim_{\theta \to 1} \limsup_{x \to \infty} \frac{Q(x)}{Q(\theta x)} = 1.$$ 

Then there is a positive constant $C$, so that if $k \geq 1$ and $\xi_{k+1} \leq F_k(a) < \xi_k$, then $\ell \geq k > \ell - C \log \ell$ and $\xi_k \sim F_k(a)$. Moreover, $C$ can be chosen independent of $a$ for $a \leq a_0$.

**Remarks.** The hypothesis on $Q$ is equivalent to the condition that $\log Q$ be slowly oscillating. The conditions imposed on $F$ and $G$ trivially imply that for $0 < x < y < A$, $0 < F(y) - F(x) < y - x$. Since $0 < G(x) < x$ and is increasing, $K$ is defined for $0 < x < \infty$ and is convex. Thus $Q = -K' = G(K)$.

We denote $F_k(a)$ by $\eta_k$.

**Proof of Theorem 1.** First, note that $F(\xi_\ell) < \xi_{\ell+1}$, for $\xi_{\ell+1} = F(\xi_\ell) - c_{\ell+1} - \xi_\ell + G(\xi_\ell) = K(\ell + 1) - K(\ell) - K'(\ell) > 0$ by the convexity of $K$. If $\eta_{j+1} = F(\eta_j) < F(\xi_\ell) < \xi_{\ell+1}$, then $\eta_{j+1} = F(\eta_j) < F(\xi_\ell) < \xi_{\ell+1}$, so that every interval $[\xi_{\ell+1}, \xi_\ell)$ contains at most one $\eta_j$.

Now suppose $\eta_{k+1} < \xi_{\ell+1} < \xi_\ell \leq \eta_k$. We will prove inductively that for $n > 0$,

$$0 < \xi_{\ell+n} - \eta_{k+n} \leq K(\ell + n) - \xi_\ell - \sum_{j=0}^{n-1} K'(\ell + j),$$

with strict inequality for $n > 1$.

To start the induction we have $F(\xi_\ell) \leq \eta_{k+1} < \xi_{\ell+1}$, so $0 < \xi_{\ell+1} - \eta_{k+1} \leq \xi_{\ell+1} - F(\xi_\ell) = K(\ell + 1) - K(\ell) - K'(\ell)$. Assuming $(**)$ for $n$, we have $\eta_{k+n} < \xi_{\ell+n}$ and so $\eta_{k+n+1} < F(\xi_{\ell+n}) < \xi_{\ell+n+1}$. Using the Lipschitz condition satisfied by $F$ and the induction hypothesis, we get

$$0 < \xi_{\ell+n+1} - \eta_{k+n+1} = \xi_{\ell+n+1} - F(\xi_{\ell+n}) + F(\xi_{\ell+n}) - F(\eta_{k+n})$$

$$< \xi_{\ell+n+1} - \xi_{\ell+n} + G(\xi_{\ell+n}) + (\xi_{\ell+n} - \eta_{k+n})$$

$$\leq K(\ell + n + 1) - K(\ell + n) - K'(\ell + n)$$

$$+ \left( K(\ell + n) - K(\ell) - \sum_{j=0}^{n-1} K'(\ell + j) \right)$$

$$= K(\ell + n + 1) - K(\ell) - \sum_{j=0}^{n} K'(\ell + j),$$

so $(**)$ is established.
For how many $n$ is it true that $\xi_{\ell+n} - \eta_{k+n} < \xi_{\ell+n} - \xi_{\ell+n+1}$? By (**), this condition will hold providing that

$$K(\ell + n) - K(\ell) - \sum_{j=0}^{n-1} K'(\ell + j) < K(\ell + n) - K(\ell + n + 1),$$
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or, equivalently,

$$\sum_{j=0}^{n-1} Q(\ell + j) < \int_{\ell}^{\ell + n + 1} Q(u) \, du.$$

Since $Q$ is decreasing, it is enough to have

$$\sum_{j=0}^{n-1} Q(\ell + j) < \sum_{j=1}^{n+1} Q(\ell + j)$$

or $Q(\ell) < Q(\ell + n) + Q(\ell + n + 1)$ and so $Q(\ell) < 2Q(\ell + n + 1)$ suffices. Now if $\theta_0 > 1$ and is sufficiently small, $Q(x)/Q(\theta_0 x) < 2$, provided $x$ is large. Picking $1 < \theta < \theta_0$ and a suitably large index $\ell'$, we obtain the fact that if $\ell + n < \theta \ell'$, then we do indeed have $\xi_{\ell+n} - \eta_{k+n} < \xi_{\ell+n} - \xi_{\ell+n+1}$.

To recapitulate, if an interval $[\ell + j, \ell + n)$ contains no $n$ and $\ell$ is large, then the intervals $[\ell + j + x, \ell + n)$ contain exactly one $n$ for $\ell + j < \ell'$. We may assume that there are infinitely many intervals $[\xi_{\ell+1}, \xi_{\ell})$ not containing an $n$. Denote by $\ell_\nu$, $\nu \geq 0$, $\ell_\nu < \ell_{\nu+1}$, those indices for which $[\xi_{\ell_{\nu+1}}, \xi_{\ell_{\nu}}]$ has no $n$ and $\ell_0 > \ell'$. By the results found earlier, we have $\ell_0 \geq \theta_0$ and, in general, $\ell_\nu \geq \theta^\nu \theta_0$. Of course, since $\eta_0 = \xi_0$, we have $\eta_k < \xi_k$, so $\ell' \geq k$. Assume first that $k \geq \ell_0$, and let $\xi_{\ell_0} \leq \eta_k < \xi_{\ell'}$. Define $\mu$ by $\ell_{\mu+1} > \ell > \ell_{\mu}$. Then $\ell = k + \mu + 1 + B$ where $B$ is the number of intervals $[\xi_{\ell_{\mu+1}}, \xi_{\ell_{\mu}}]$, $1 \leq \mu \leq \ell'$, which contain no $n$. Hence we have

$$\log \ell > \log \ell_{\mu} > \mu \log \theta,$$

and $k > \ell' - (\log \theta)^{-1} \log \ell' - (\ell' + 1)$. By considering separately the case $\ell' \leq k < \ell_0$ for which $k = \ell - B \geq \ell - \ell'$ and the case $k \leq \ell'$, we easily obtain the first assertion of the theorem.

To check the uniformity of $C$, let $b < a$. $H_b(x)$ is defined by $H_b(x) = \int_x^b \frac{du}{Q(u)}$, and $K_b$ and $Q_b$ are defined in the obvious way. For $x > 0$, $Q_b(x) = Q(x + H(b))$ and

$$\frac{Q_b(x)}{Q_b(\theta_0 x)} = \frac{Q(x + H(b))}{Q(\theta_0 x + H(b))} < \frac{Q(x + H(b))}{Q(\theta_0 (x + H(b))} < 2$$

if $x + H(b) > \ell'$. Since $C$ is determined by $\theta_0$ and $\ell'$, uniformity is proved.

Finally, to check that $\xi_k \sim \eta_k$, note that $K$ satisfies the same slowly oscillating condition as $Q$. If $\varepsilon$ is small and $\theta > 1$ is suitably chosen, we get from $K(x) = \int_x^\infty Q(u) \, du$ the inequalities

$$K(x) < (1 + \varepsilon) \int_x^\infty Q(\theta u) \, du = \frac{(1 + \varepsilon)}{\theta} \int_{\theta x}^\infty Q(u) \, du < (1 + \varepsilon)K(\theta x)$$
provided $x$ is large. Now if $\xi_{r+1} \leq \eta_k < \xi_r$, then for large $k$, $\ell + 1 < k + C' \log k$, for any $C' > C$, and

$$0 < 1 - \left( \frac{\eta_k}{\xi_k} \right) \leq 1 - \left( \frac{\xi_{r+1}}{\xi_k} \right)$$

$$= 1 - \frac{(\ell + 1)/K(k)}{K(k)}$$

$$< 1 - \frac{(k + C' \log k)/K(k)}{K(k)}.$$  

But $(K(\ell k)/K(k)) < \varepsilon$ if $\ell$ is suitably chosen, and the result follows.

As we mentioned, under the generality of our assumptions, we cannot expect a true second asymptotic term. Indeed, we have obtained only a nontrivial lower bound for $\eta_k/\xi_k$. Nevertheless, as crude as our method is, it seems to produce an error of the correct order of magnitude. For example, for $F(x) = \sin x$, it is known that

$$F_k(a) = (3/k)^{1/2} \left\{ 1 - \frac{3 \log k}{10k} + o \left( \frac{\log k}{k} \right) \right\},$$

(see [1]) if $a$ is small. If we set $G(x) = Bx^n(1 + o(x))$ and assume that $0 < G(y) - G(x) < y - x$ for $0 < x < y < A$, and we also assume for simplicity that $n > 2$, then a simple calculation gives $K(x) = (B(n - 1)x)^{1/1-n} \{1 + o(1/x)\}$. It follows that $Q$ satisfies the hypothesis of Theorem 1. Hence,

$$(B(n - 1)x)^{1/1-n} \eta_k > (B(n - 1)x)^{1/1-n} \{1 - D(\log k/k)\}$$

for suitable $D$.

Finally, for $G(x) = x^2 \exp(-1/x)$, another routine computation gives

$$\frac{1}{\log k} > \eta_k > \frac{1}{\log k} \left\{ 1 - \frac{C}{k} \right\}.$$ 

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