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MOSCO CONVERGENCE AND REFLEXIVITY

GERALD BEER AND JONATHAN M. BORWEIN

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ABSTRACT. In this note we aim to show conclusively that Mosco convergence of convex sets and functions and the associated Mosco topology τ_M are useful notions only in the reflexive setting. Specifically, we prove that each of the following conditions is necessary and sufficient for a Banach space X to be reflexive: (1) whenever $A, A_1, A_2, A_3, \ldots$ are nonempty closed convex subsets of X with $A = \tau_M - \lim A_n$, then $A^\circ = \tau_M - \lim A_n^\circ$; (2) τ_M is a Hausdorff topology on the nonempty closed convex subsets of X; (3) the arg min multifunction $f \rightrightarrows \{x \in X: f(x) = \inf_X f\}$ on the proper lower semicontinuous convex functions on X, equipped with τ_M, has closed graph.

1. INTRODUCTION

Let $\mathscr{C}(X)$ be the nonempty closed convex subsets of a Banach space X. A well-studied notion of convergence of sequences of convex sets in reflexive spaces, from the point of view of both optimization and Banach space geometry [At, BP, Be1, Be2, BF, LP, Mc, Mo1, Mo2, SW, So, Ts], is that of Mosco convergence. Specifically, a sequence $\langle A_n \rangle$ in $\mathscr{C}(X)$ is declared *Mosco convergent* to $A \in \mathscr{C}(X)$ provided both of the following conditions hold:

(Mi) for each $a \in A$ there exists a sequence $\langle a_n \rangle$ convergent in norm to a such that for each n, $a_n \in A_n$;

(Mii) whenever $n(1) < n(2) < \cdots$ and $a_k \in A_{n(k)}$ for each $k \in Z^+$, then the weak $(\sigma(X, X^*))$ convergence of $\langle a_k \rangle$ to $x \in X$ implies $x \in A$.

Perhaps the most important feature of Mosco convergence in reflexive spaces is its *stability with respect to duality*, as expressed by the (sequential) continuity of the polar map $A \to A^\circ$, or by the (sequential) continuity of the conjugate map $f \to f^*$ defined on $\Gamma(X)$, the proper lower semicontinuous convex functions X, where functions are identified with their epigraphs. These results are due to Mosco in [Mo2], although they were first obtained in finite

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dimensions by Wijsman [Wi]. In [Be1] a simple Vietoris-type topology τ_M compatible with Mosco convergence of sequences in any Banach space was introduced, and in [Be2], Mosco's sequential continuity results were extended to topological continuity theorems with respect to $\langle \mathscr{C}(X), \tau_M \rangle$, and $\langle \Gamma(X), \tau_M \rangle$, respectively (for reflexive X, τ_M is first countable if and only if X is separable, so that sequences do not otherwise determine the topology).

The above results—and all other positive results of any depth regarding Mosco convergence—invariably assume reflexivity of the underlying space. We show in this paper that reflexivity assumptions are indeed required. As a main result, we show that even sequential continuity of the polar map must fail without reflexivity.

2. Notation and some background material

Unless otherwise stated, X will denote a Banach space with origin θ and closed unit ball U. The continuous dual of X will be denoted by X^*. For each $A \in \mathscr{C}(X)$ its *polar* A° is the following weak star $(\sigma(X^*, X))$ closed subset of X^*: $A^\circ \equiv \{y \in X^*: y(a) \leq 1 \text{ for each } a \in A\}$. The *epigraph* of an extended real valued function f on X, as a subset of $X \times R$, is defined by

$$\operatorname{epi} f \equiv \{(x, \alpha): x \in X, \alpha \in R, \text{ and } \alpha \geq f(x)\}.$$

The function is called *proper* if $\operatorname{epi} f$ is nonempty and contains no vertical line. We denote the *sublevel set* $\{x \in X: f(x) \leq \alpha\}$ at height α of f by $\operatorname{lev}(f; \alpha)$. For each $f \in \Gamma(X)$, its *conjugate* $f^* \in \Gamma(X^*)$ is defined at each $y \in X^*$ by $f^*(y) = \sup\{y(x) - f(x): x \in X\}$. The condition $(y, \alpha) \in \operatorname{epi} f^*$ means that f majorizes $x \to y(x) - \alpha$. By $\arg\min f$, we mean the (possibly empty) set of minimizers of f: $\arg\min f \equiv \{x \in X: f(x) = \inf_X f\}$.

To describe the topology of Mosco convergence τ_M on $\mathscr{C}(X)$, we need some standard notation from the theory of hyperspaces (topologies on sets of subsets). If $E \subset X$, we define subsets E^- and E^+ of $\mathscr{C}(X)$ as follows:

$$E^- \equiv \{A \in \mathscr{C}(X): A \cap E \neq \varnothing\} \qquad E^+ \equiv \{A \in \mathscr{C}(X): A \subset E\}.$$

The *Mosco topology* τ_M [Be1] on $\mathscr{C}(X)$ has as subbase all sets of the form V^- where V is strongly open, and $(K^C)^+$ where K is weakly compact. Since $((K_1 \cup K_2)^C)^+ = (K_1^C)^+ \cap (K_2^C)^+$, all sets of the following form determine a base for the topology:

$$[V_1, V_2, \ldots, V_n; K] \equiv \left(\bigcap_{i=1}^n V_i^-\right) \cap (K^C)^+,$$

where $V_1, V_2, \ldots, V_n$ are strongly open, and K is weakly compact. Since elements of $\Gamma(X)$ may be viewed as nonempty closed convex subsets of $X \times R$, we may consider the function space as a topological subspace of $\langle \mathscr{C}(X \times R), \tau_M \rangle$. It is in this sense that $\langle \Gamma(X), \tau_M \rangle$ is to be understood. Of course, Mosco convergence of sequences of convex functions may also be defined directly (see Lemma 1.10 of [Mo1] or [At, p. 297]).

3. Polarity is not continuous unless X is reflexive

As noted in the introduction, if X is reflexive, then the polar map from $\langle \mathscr{C}(X), \tau_M \rangle$ to $\langle \mathscr{C}(X^*), \tau_M \rangle$ is continuous. Here, we show that the converse holds. The main idea is this: If X is not reflexive, there is a norm one element z of X^* that is not norm attaining on the unit ball. From this, we construct a sequence $\langle C_n \rangle$ in $\mathscr{C}(X)$ Mosco convergent to $C = \{\theta\}$ such that the points of X^* that are strong limits of sequences of points chosen from $\langle C_n^\circ \rangle$ are constrained to lie in a closed halfspace determined by a Hahn-Banach functional for z. Thus, $\langle C_n^\circ \rangle$ cannot converge to $C^\circ = X^*$ in the sense of Mosco.

A preparatory lemma will be helpful.

Lemma 3.1. *Let $\langle A_n \rangle$ be a decreasing sequence of nonempty closed and bounded convex subsets of X with empty intersection. If for each n, $C_n = \mathrm{co}(\{\theta\} \cup A_n)$, then $C_n \in \mathscr{C}(X)$ and $\bigcap_{n \geq 1} C_n = \{\theta\}$.*

Proof. Clearly, $\theta \in \bigcap_{n \geq 1} C_n$. Suppose $c \in \bigcap_{n \geq 1} C_n$ and $c \neq \theta$. For each n, $c = \lambda_n a_n$ where $\lambda_n \in [0, 1]$ and $a_n \in A_n$. By passing to a subsequence we may assume that $\langle \lambda_n \rangle$ converges to λ. Clearly, $\lambda \neq 0$, or else c would be the origin, because $\langle a_n \rangle$ is uniformly bounded. But then $c/\lambda = \lim_{n \to \infty} c/\lambda_n = \lim_{n \to \infty} a_n$, yielding $\bigcap_{n \geq 1} A_n \neq \varnothing$. Thus, $\bigcap_{n \geq 1} C_n = \{\theta\}$. □

Theorem 3.2. *Suppose X is a Banach space such that whenever $C, C_1, C_2, \ldots$ are closed and bounded convex sets with $C = \tau_M - \lim C_n$, then $C^\circ = \tau_M - \lim C_n^\circ$. Then X is reflexive.*

Proof. Suppose X is not reflexive. By James's theorem, there is a norm one functional $z \in X^*$ that does not attain its norm on the closed unit ball (see [Da, p. 63] or [Ja]). Consider for $n \in Z^+$ the closed bounded convex set

$$A_n \equiv \{x \in X : z(x) \geq 1 - 2^{-n} \text{ and } \|x\| \leq 1\},$$

and let

$$C_n \equiv \mathrm{co}(\{\theta\} \cup A_n) = \{\alpha x : x \in A_n \text{ and } 0 \leq \alpha \leq 1\}.$$

We will show: (a) $\tau_M - \lim C_n = C = \{\theta\}$, but that (b) $\tau_M - \lim C_n^\circ \neq C^\circ = X^*$ (we make no claim that $\tau_M - \lim C_n^\circ$ exists).

(a) Since $\theta \in C_n$ for each n, (Mi) holds. To show (Mii) holds, suppose that $c_k \in C_{n(k)}$ for $k = 1, 2, 3, \ldots$, and $\langle c_k \rangle$ converges weakly to x. Since $\langle C_n \rangle$ is a decreasing sequence of weakly closed sets, we get $x \in C_n$ for each n. Thus, by Lemma 3.1, $x = \theta \in C$ as required.

(b) Since z has unit norm in X^*, there is an element F of X^{**} such that $F(z) = \|F\| = 1$. Suppose for each n, $y_n \in C_n^\circ$ and $\langle y_n \rangle$ converges to y_0 in norm. We will show that $F(y_0) \leq 1$, so that condition (Mi) fails for any point of $C^\circ = X^*$ that fails to lie in the closed half space $\{y \in X^* : F(y) \leq 1\}$.

Consider this weak neighborhood of F defined by

$$W_n \equiv \{G \in X^{**} : |F(z) - G(z)| < 2^{-n} \text{ and } |F(y_0) - G(y_0)| < 2^{-n}\}.$$

By Goldstein's density theorem [Da, p. 47], for each n, there is a point $x_n \in X$ (appropriately identified as a subspace of X^{**}) with $x_n \in W_n$ and with $\|x_n\| \leq 1$. In particular, $z(x_n) \geq 1 - 2^{-n}$ so that $x_n \in C_n$. Since $y_n \in C_n^\circ$, we have $y_n(x_n) \leq 1$. From the definition of W again, $|F(y_0) - y_0(x_n)| < 2^{-n}$. We obtain

$$\begin{aligned} F(y_0) \leq y_0(x_n) + 2^{-n} &\leq y_n(x_n) + \|y_n - y_0\| \cdot \|x_n\| + 2^{-n} \\ &\leq 1 + \|y_n - y_0\| + 2^{-n}. \end{aligned}$$

As $\langle y_n \rangle$ is convergent in norm to y_0, we get $F(y_0) \leq 1$, as desired. □

We note that in [Be4], an example is given showing that τ_M-continuity of $C \to C^\circ$ fails for sequences of hyperplanes in the nonreflexive space l_1. The construction is completely unrelated to the one presented above. Indeed there are nonreflexive spaces, such as c_0, for which τ_M-continuity of $C \to C^\circ$ holds for sequences of hyperplanes.

From Theorem 3.2, it easily follows that the conjugate operator fails to be sequentially τ_M-continuous in nonreflexive spaces.

Theorem 3.3. *Suppose X is a Banach space such that whenever $f, f_1, f_2, \ldots$ are proper lower semicontinuous convex functions on X with $f = \tau_M - \lim f_n$, then $f^* = \tau_M - \lim f_n^*$. Then X is reflexive.*

Proof. Let $\delta(\cdot, A)$ be the indicator function of a closed convex set A. As is well known, $\delta^*(\cdot, A)$ is the support function for A, $A^\circ = \operatorname{lev}(\delta^*(\cdot, A); 1)$ [Ho, §14]. Suppose X is not reflexive, and the conjugate map is sequentially continuous. Let C and $\langle C_n \rangle$ be as in the proof of Theorem 3.2. Then $\delta(\cdot, C) = \tau_M - \lim \delta(\cdot, C_n)$ so that $\delta^*(\cdot, C) = \tau_M - \lim \delta^*(\cdot, C_n)$. But since Mosco convergence of convex functions ensures Mosco convergence of sublevel sets at a fixed height exceeding the infimum of the limit [Be2, Lemma 3.2], we recover $C^\circ = \tau_M - \lim C_n^\circ$. This is a contradiction to Theorem 3.2. □

4. τ_M IS NOT HAUSDORFF UNLESS X IS REFLEXIVE

If X is a reflexive Banach space, it is immediate that $\langle \mathscr{C}(X), \tau_M \rangle$ is Hausdorff. To see this, if $A \in \mathscr{C}(X)$ and $C \in \mathscr{C}(X)$, and $A \cap C^C \neq \varnothing$, then there exists $a \in A$ and $\varepsilon > 0$ such that $(a + \varepsilon U) \cap C = \varnothing$. Clearly, $(a + \operatorname{int} \varepsilon U)^-$ and $((a + \varepsilon U)^C)^+$ are disjoint τ_M-open neighborhoods of A and C, respectively. We show that when X is nonreflexive, then each τ_M-open set is actually dense in $\langle \mathscr{C}(X), \tau_M \rangle$. This falls out of a geometric lemma regarding the structure of weakly compact sets in a nonreflexive space.

Lemma 4.1. *Let X be a nonreflexive space. Let K be a weakly compact subset of X and let $V_1, V_2, \ldots, V_n$ be nonempty open subsets of X (not necessarily distinct). Then there exist distinct points $x_i \in V_i$ such that* $\operatorname{co}\{x_1, x_2, \ldots, x_n\} \cap K = \varnothing$.

Proof. We rely on the fact that a weakly compact set in a nonreflexive space must have empty interior. Choose $\varepsilon > 0$ and distinct $x_i \in V_i$ such $U_i \equiv x_i + \varepsilon U \subset V_i$. For each $u \in U$, set $P(u) \equiv \operatorname{co}\{x_1 + \varepsilon u, x_2 + \varepsilon u, \ldots, x_n + \varepsilon u\}$,

so that $P(u) = P(\theta) + \varepsilon u$. We claim that for some point $u \in U$, $P(u) \cap K = \varnothing$. Else, for each $u \in U$, $[P(\theta) + \varepsilon u] \cap K \neq \varnothing$, and $\varepsilon U \subset K - P(\theta)$. This last sum is weakly compact with nonempty norm interior. We conclude that, for some u in U, $\mathrm{co}\{x_1 + \varepsilon u, x_2 + \varepsilon u, \dots, x_n + \varepsilon u\} \cap K = \varnothing$. □

Theorem 4.2. *Let X be a nonreflexive Banach space. Then neither $\langle \mathscr{C}(X), \tau_M \rangle$ nor $\langle \Gamma(X), \tau_M \rangle$ is Hausdorff.*

Proof. Let $[V_1, V_2, \dots, V_m; K]$ and $[W_1, W_2, \dots, W_n; K']$ be basic τ_M-open subsets of $\mathscr{C}(X)$. By Lemma 4.1 there exists $x_i \in V_i$ and $w_i \in W_i$ such that $\mathrm{co}\{x_1, x_2, \dots, x_m, w_1, w_2, \dots, w_n\} \cap (K \cup K') = \varnothing$. Clearly, the polytope $P \equiv \mathrm{co}\{x_1, x_2, \dots, x_m, w_1, w_2, \dots, w_n\}$ lies in the intersection of $[V_1, V_2, \dots, V_m; K]$ and $[W_1, W_2, \dots, W_n; K']$. Thus, no two τ_M-open subsets of $\mathscr{C}(X)$ can be disjoint. From this, $\langle \Gamma(X), \tau_M \rangle$ must fail to be Hausdorff because $A \to \delta(\cdot, A)$ is an embedding of $\langle \mathscr{C}(X), \tau_M \rangle$ into $\langle \Gamma(X), \tau_M \rangle$. □

5. On the arg min multifunction

From the perspective of convex optimization, a minimal requirement that a topology τ on $\Gamma(X)$ should satisfy to be of *any* interest is the following: if x_λ is a minimizer of f_λ and $f_0 = \tau - \lim f_\lambda$ and $\|x_\lambda - x_0\| \to 0$, then x_0 should be a minimizer of f_0. When $\tau = \tau_M$, we show that this condition is satisfied if and only if X is reflexive.

The condition expressed in the preceding paragraph is conveniently described in the language of multifunctions. By a *multifunction* $\varphi: T \rightrightarrows X$ from a topological space T to another topological space X, we mean a function that assigns to each $t \in T$ a possibly empty closed subset $\varphi(t)$ of X. The *graph* of φ is this subset of $T \times X$: $\{(t, x): x \in \varphi(t)\}$. When X is a Banach space, we say that φ has *strongly* (resp. *weakly*) *closed graph* provided its graph is a closed subset of $T \times X$ when X is equipped with the norm (resp. weak) topology. We have claimed that the arg min multifunction has strongly closed graph if and only if X is reflexive. Remarkably, the multifunction fails to have weakly closed graph, even though this is true sequentially! Lemma 4.1 again plays a role in the analysis.

Theorem 5.1. *Let X be a Banach space. The following are equivalent*:

(a) *X is reflexive*;
(b) $\arg\min: \langle \Gamma(X), \tau_M \rangle \rightrightarrows X$ *has strongly closed graph*;
(c) *whenever $f_0 = \tau_M - \lim f_\lambda$ and $x_0 = \sigma(X, X^*) - \lim x_\lambda$ where $x_\lambda \in \arg\min f_\lambda$ and $\langle x_\lambda \rangle$ is eventually bounded, then $x_0 \in \arg\min f_0$.*

Proof. (a) ⇒ (c). By passing to a subnet we may assume that for some $\rho > 0$ and all λ, we have $\|x_\lambda\| \leq \rho$. Suppose to the contrary that $x_0 \notin \arg\min f_0$. Choose $x_1 \in X$ with $f_0(x_1) < f_0(x_0)$. Since f_0 is the supremum of the continuous affine functionals that it majorizes, there exists $(y, \beta) \in X^* \times R$ with $\beta > f_0^*(y)$ such that $y(x_0) - \beta > f_0(x_1)$. Choose β_0 strictly between $y(x_0) - \beta$

and $f_0(x_1)$. Then $H \equiv \{x \in X : y(x) \geq \beta_0 + \beta\}$ is a weak neighborhood of x_0. By reflexivity, the following half-cylindrical slice of the graph of $x \to y(x) - \beta$ is a weakly compact subset of $X \times R$:

$$K \equiv \{(x, \alpha) : \|x\| \leq \rho, x \in H \text{ and } \alpha = y(x) - \beta\}.$$

In fact, K is the graph of a weakly continuous function restricted to a weakly compact set. The condition $\beta > f_0^*(y)$ guarantees that $f_0 \in (K^C)^+$. Also, $f_0 \in U^-$ where $U \equiv X \times (-\infty, \beta_0)$ because $f_0(x_1) < \beta_0$. Take λ_0 in the underlying directed set for the net such that for all $\lambda \geq \lambda_0$, we have both $f_\lambda \in U^- \cap (K^C)^+$ and $x_\lambda \in H$. Now $\operatorname{epi} f_\lambda \cap K = \varnothing$ can occur only if $\operatorname{epi} f_\lambda$ lies above K, for epigraphs recede in the vertical direction. As a result, for all $\lambda \geq \lambda_0$, we have

$$f_\lambda(x_\lambda) > y(x_\lambda) - \beta \geq (\beta_0 + \beta) - \beta = \beta_0.$$

However, the condition $f_\lambda \in U^-$ means that $\inf_X f_\lambda < \beta_0$. This contradicts our assumption that $x_\lambda \in \arg\min f_\lambda$.

(c) $\Rightarrow$ (b). This is obvious, for if a net $\langle x_\lambda \rangle$ in X is convergent strongly, then it is eventually norm bounded.

(b) $\Rightarrow$ (a). Suppose X is not reflexive. We show that

$$\mathscr{A} \equiv \{(g, x) : g \in \Gamma(X) \text{ and } \{x\} = \arg\min g\}$$

is dense in $\Gamma(X) \times X$, where $\Gamma(X)$ is equipped with the Mosco topology and X is equipped with the norm topology. Let (f, a) be an arbitrary point in the product, and let $[V_1, \ldots, V_n; K]$ be an arbitrary τ_M-neighborhood of f, and let $\varepsilon > 0$. By Lemma 4.1, we can choose distinct points $x_i \in \pi_X(V_i)$ and $x_0 \in \operatorname{int}(a + \varepsilon U)$ such that $\operatorname{co}\{x_0, x_1, \ldots, x_n\} \cap \pi_X(K) = \varnothing$. Let $P = \operatorname{co}\{x_0, x_1, \ldots, x_n\}$. Since P and $\pi_X(K)$ lie a positive distance apart and $\pi_X(V_i)$ is open for each i, there is no loss in generality in assuming that $\{x_0, x_1, \ldots, x_n\}$ is an affinely independent set. Thus, each point of P may be uniquely represented as a convex combination of the vertices. For each $i \in \{1, 2, \ldots, n\}$, pick α_i with $(x_i, \alpha_i) \in V_i$, and take $\alpha_0 < \min\{\alpha_i : 1 \leq i \leq n\}$. Then $g \in \Gamma(X)$ defined by

$$g(x) = \begin{cases} \Sigma\lambda_i\alpha_i & \text{if } x = \Sigma\lambda_i x_i \text{ with } \lambda_i \geq 0 \text{ and } \Sigma\lambda_i = 1, \\ \infty & \text{otherwise}, \end{cases}$$

is well defined and, by construction, lies in $[V_1, \ldots, V_n; K]$. Moreover, $\arg\min g = \{x_0\}$. Thus, $(g, x_0) \in \mathscr{A} \cap [V_1, \ldots, V_n; K] \times (a + \varepsilon U)$, establishing density of $\mathscr{A}$ in $\Gamma(X) \times X$. As a result, the graph of the arg min multifunction is not strongly closed. □

Without reflexivity, it is a routine exercise to check that arg min has *weakly sequentially closed graph*: If x_n is a minimizer of f_n, and $x = \alpha(X, X^*) - \lim x_n$ and $f = \tau_M - \lim f_n$, then x must be a minimizer of f. Combining this with the equivalence of (a) and (b) in Theorem 5.1, we see that reflexivity of X is necessary for first countability of $\langle \Gamma(X), \tau_M \rangle$. Taking note of the results

of [Be1], we see that first countability of the function space for separable X is equivalent to reflexivity of X.

Condition (c) in the statement Theorem 5.1 is formally weaker than weak closedness of the graph of arg min: $\langle \Gamma(X), \tau_M \rangle \rightrightarrows X$. We intend to show that weak closedness actually fails unless X is finite dimensional. What goes wrong in infinite dimensions is that a weakly convergent net of minimizers of a Mosco convergent net of convex functions need not be eventually bounded. For this purpose, we present a characterization of τ_M-convergence of nets, generalizing the sequential conditions (Mi) and (Mii), the essence of whose proof may be found in the proof of Theorem 5.2 of [Be3].

Lemma 5.2. *Let X be a reflexive Banach space, and let $\langle A_\lambda \rangle_{\lambda \in \Lambda}$ be a net in $\mathscr{C}(X)$. Then $A = \tau_M - \lim_{\lambda \in \Lambda} A_\lambda$ if and only if both of the following conditions are satisfied*:

(M1) *For each $a \in A$, each norm open neighborhood of a meets $\langle A_\lambda \rangle$ eventually; and*

(M2) *whenever $\Lambda^\#$ is a cofinal subset of Λ and for each $\lambda \in \Lambda^\#$, we have $a_\lambda \in A_\lambda$, and $\langle a_\lambda \rangle_{\lambda \in \Lambda^\#}$ is norm bounded, then A contains each weak cluster point of $\langle a_\lambda \rangle_{\lambda \in \Lambda^\#}$.*

Proof. Evidently, (M1) is equivalent to the requirement that whenever V is norm open and $A \in V^-$, then $\langle A_\lambda \rangle_{\lambda \in \Lambda}$ is in V^- eventually.

We show that (M2) holds if and only if for each weakly compact subset K of X, whenever $\langle A_\lambda \rangle_{\lambda \in \Lambda}$ is in K^- frequently, then $A \in K^- = [(K^C)^+]^C$. Suppose first that (M2) holds, and for some cofinal subset $\Lambda^\#$ of Λ and for each $\lambda \in \Lambda^\#$, we have $A_\lambda \cap K \neq \varnothing$. Pick a_λ in the intersection. Since K is weakly compact, $\langle a_\lambda \rangle_{\lambda \in \Lambda^\#}$ has a weak cluster point $x_0 \in K$; so, by (M2), we have $x_0 \in A$. Thus, $A \in K^-$. Conversely, suppose (M2) fails. Then there exists a cofinal subset $\Lambda^\#$ and a norm bounded net $\langle a_\lambda \rangle_{\lambda \in \Lambda^\#}$ that has a weak cluster point $x_0 \in A^C$ such that for each $\lambda \in \Lambda^\#$, we have $a_\lambda \in A_\lambda$. Choose $y \in X^*$ and $\alpha \in R$ such that $y(x_0) < \alpha < \inf\{y(a) : a \in A\}$. Choose $\rho > 0$ such that for each $\lambda \in \Lambda^\#$, we have $\|a_\lambda\| \leq \rho$. Then by reflexivity, $K \equiv \{x \in X : y(x) \leq \alpha\}$ and $\|x\| \leq \rho\}$ is weakly compact, and since x_0 is a weak cluster point of $\langle a_\lambda \rangle_{\lambda \in \Lambda^\#}$, K contains a_λ frequently. Thus, $A_\lambda \in K^-$ frequently, whereas $A \notin K^-$. □

We note that Lemma 5.2 could have been used to establish (a) ⇒ (c) in the proof of Theorem 5.1, but the geometric proof that we presented seems more informative.

Theorem 5.3. *Let X be a reflexive Banach space. Then the graph of* arg min: $\langle \Gamma(X), \tau_M \rangle \rightrightarrows X$ *is weakly closed if and only if X is finite dimensional.*

Proof. If X is finite dimensional, then the graph is weakly closed because it is strongly closed for any reflexive space.

Now suppose that X is infinite dimensional. Let $x_0 \in X$ be a norm one vector, and define $f \in \Gamma(X)$ by

$$f(x) = \begin{cases} -\alpha & \text{if } x = \alpha x_0 \text{ with } 0 \le \alpha \le 1 \\ \infty & \text{otherwise.} \end{cases}$$

Notice that $\theta \notin \arg\min f$. We produce a net $\langle f_\lambda \rangle$ of proper lower semicontinuous convex functions τ_M-convergent to f such that for each λ, $\arg\min f_\lambda$ is a singleton $\{x_\lambda\}$ and such that $\theta = \sigma(X, X^*) - \lim x_\lambda$.

Let $\mathscr{F}$ be the collection of finite subsets of $X^* - \{\theta\}$, partially ordered by inclusion. Equip $\Lambda \equiv \mathscr{F} \times Z^+$ with the product order, and define a net $\langle x_\lambda \rangle_{\lambda \in \Lambda}$ as follows: if $\lambda = (\{y_1, y_2, \dots, y_n\}, k)$, let x_λ be a point of $\{x \in X : |y_i(x)| < 1/k$ for $i = 1, \dots, n\}$, such that $\{\theta, x_0, x_\lambda\}$ forms an affinely independent set and $\|x_\lambda\| > k$. Clearly, $\theta = \sigma(X, X^*) - \lim x_\lambda$ and at the same time, $\lim_\lambda \|x_\lambda\| = \infty$. We now define f_λ as follows: If x is a convex combination of $\{\theta, x_0, x_\lambda\}$, say $x = \alpha_1\theta + \alpha_2 x_0 + \alpha_3 x_\lambda$, then $f_\lambda(x) = \alpha_2(-1 + 1/k) - \alpha_3$. Otherwise, we set $f_\lambda(x) = \infty$. Notice that $x = x_\lambda$ is the unique minimizer of f_λ, and $f_\lambda(x_\lambda) = -1$.

To show that $f = \tau_M - \lim f_\lambda$, we show that the conditions (M1) and (M2) of Lemma 5.2 are both satisfied with respect to epigraphs. Since the graph of f consists of $\operatorname{co}(\{(\theta, 0), (x_0, -1)\})$ and the graph of f_λ contains $\operatorname{co}(\{(\theta, 0), (x_0, -1 + 1/k)\})$, where k is the second coordinate of λ, it is clear that (M1) is satisfied. For (M2), suppose $\Lambda^\#$ is a cofinal subset of Λ, $\langle (w_\lambda, \beta_\lambda) \rangle_{\lambda \in \Lambda^\#}$ is a uniformly bounded net such that for each $\lambda \in \Lambda^\#$, $(w_\lambda, \beta_\lambda) \in \operatorname{epi} f_\lambda$, and $\langle (w_\lambda, \beta_\lambda) \rangle_{\lambda \in \Lambda^\#}$ has (w, β) as a weak cluster point. We may write

$$w_\lambda = \mu_\lambda x_\lambda + (1 - \mu_\lambda)(\alpha_\lambda x_0),$$

where $\mu_\lambda \in [0, 1]$ and $\alpha_\lambda \in [0, 1]$. By passing to a subnet, we may assume that $\langle (w_\lambda, \beta_\lambda) \rangle_{\lambda \in \Lambda^\#}$ converges weakly to (w, β), that $\langle \mu_\lambda \rangle_{\lambda \in \Lambda^\#} \to \mu_0$, and that $\langle \alpha_\lambda \rangle_{\lambda \in \Lambda^\#} \to \alpha_0$. Since $\langle w_\lambda \rangle_{\lambda \in \Lambda^\#}$ and $\langle \alpha_\lambda \rangle_{\lambda \in \Lambda^\#}$ are uniformly bounded and $\lim_{\lambda \in \Lambda^\#} \|x_\lambda\| \to \infty$, we conclude that $\mu_0 = 0$. Since addition and scalar multiplication are jointly continuous, we have

$$w = \sigma(X, X^*) - \lim w_\lambda = 0 \cdot \theta + 1 \cdot \alpha_0 x_0 = \alpha_0 x_0.$$

Since $\beta_\lambda \ge f_\lambda(w_\lambda)$ for each $\lambda \in \Lambda^\#$, it follows that

$$\begin{aligned} \beta = \lim_{\lambda \in \Lambda^\#} \beta_\lambda &\ge \limsup_{\lambda \in \Lambda^\#} f_\lambda(\mu_\lambda x_\lambda + (1 - \mu_\lambda)(\alpha_\lambda x_0)) \\ &= \limsup_{\lambda \in \Lambda^\#} \mu_\lambda f_\lambda(x_\lambda) + \alpha_\lambda(1 - \mu_\lambda)(f_\lambda(x_0)) \\ &= 0(-1) + \alpha_0 \cdot 1 \cdot (-1) = -\alpha_0 = f(w). \end{aligned}$$

This proves that $(w, \beta) \in \operatorname{epi} f$. Thus, (M2) is verified and $f = \tau_M - \lim f_\lambda$. Since (f_λ, x_λ) is in the graph of arg min for each λ and (f, θ) is not in the graph of arg min, the multifunction fails to have weakly closed graph. □

In [Be3], a net in $\langle A_\lambda \rangle$ in $\mathscr{C}(X)$ was called *topologically Mosco convergent* to $A \in \mathscr{C}(X)$ provided both of the following conditions are met: (a) for each $a \in A$, each strong (norm) neighborhood of a meets $\langle A_\lambda \rangle$ eventually, and (b) if each weak neighborhood of a point $x \in X$ meets $\langle A_\lambda \rangle$ frequently, then $x \in A$. This parallels the usual definition of topological convergence of a net of sets in a topological space (see [KT, p. 24]). In view of Lemma 5.2, the conjunction of (a) and (b) is formally stronger than $A = \tau_M - \lim A_\lambda$. The last example shows that (a) and (b) together are in fact stronger, because each weak neighborhood of $(\theta, -1)$ meets $\langle \operatorname{epi} f_\lambda \rangle$ eventually, whereas $(\theta, -1) \notin \operatorname{epi} f$.

The thrust of this article is that Mosco convergence is probably not worthy of study outside the reflexive setting. An alternative to τ_M for general Banach spaces is the topology of uniform convergence of distance functions on bounded subsets of X, which is a stronger completely metrizable topology on $\mathscr{C}(X)$ for any Banach space X [ALW], and which is stable with respect to duality, without reflexivity, or even completeness [Be4].

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DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE UNIVERSITY, LOS ANGELES, LOS ANGELES, CALIFORNIA 90032

DEPARTMENT OF MATHEMATICS, STATISTICS AND COMPUTING SCIENCE, DALHOUSIE UNIVERSITY, HALIFAX, NOVA SCOTIA B3H 3J5 CANADA