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A CHARACTERIZATION OF CONTINUA THAT CONTAIN NO n-ODS AND NO W-SETS

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ABSTRACT. A proper, nondegenerate subcontinuum K of a continuum Y is said to be a W-set if, for every continuum X and map f of X onto Y, some subcontinuum of X is mapped by f onto K. Jim Davis asked whether a simple closed curve is the only atriodic continuum that contains no W-set. An affirmative answer is given to this question. The result follows as a corollary to the more general theorem that a continuum contains no n-od and has no W-set if and only if it is a graph in which every point is contained in a simple closed curve. Properties of this class of graphs are also described.

Van Nall [3, p. 192] has proved that a decomposable continuum is a simple closed curve if and only if it is atriodic and contains no W-sets. In this paper, "decomposable" is removed from the hypothesis by proving that an atriodic, indecomposable continuum contains a W-set. This result follows from part of the proof of the main theorem that characterizes a continuum that contains no n-ods, for some integer n, and has no W-sets as a graph in which every point is contained in a simple closed curve.

DEFINITIONS AND PRELIMINARIES

A *continuum* is a compact, connected metric space. A proper, nondegenerate subcontinuum K of a continuum Y is a *W-set* in Y if, for every continuum X and map f of X onto Y, some subcontinuum of X is mapped by f onto K. A continuum belongs in *class W* if every proper, nondegenerate subcontinuum is a W-set. A *graph* is a continuum that is the union of a finite number of arcs and contains only a finite number of points of order different from 2. A continuum L is an *n-od*, $n \geq 2$, if L contains a subcontinuum H, called the *hub*, whose complement has at least n components. A continuum X is called a *θ_n-continuum*, $n \geq 1$, if no subcontinuum of X separates it into more

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than n components. Note that a continuum that contains no n-ods must be a θ_{n-1}-continuum. Also, if $x \in X$ and H_1 and H_2 are disjoint sets in X, x *cuts* H_1 from H_2 if every subcontinuum of X that intersects H_1 and H_2 contains x.

Suppose f is a map from a continuum X onto a continuum Y, H is a closed subset of Y, and $y \in Y \backslash H$. Let U_1, $U_2, \ldots$ be a nested sequence of open sets in X such that $f^{-1}(H) = \bigcap_{i=1}^{\infty} U_i$ where $U_1 \cap f^{-1}(y) = \phi$, and let x be an element of $f^{-1}(y)$. For each i, let C_i be the component of $X \backslash U_i$ that contains x. For each i, let $C^i_{y,H} = f(C_i)$ and observe that $C^i_{y,H} \subset C^j_{y,H}$ if $i < j$. Let $C_{y,H} = \mathrm{cl}(\bigcup_{i=1}^{\infty} C^i_{y,H}) = f(\mathrm{cl}(\bigcup_{i=1}^{\infty} C_i))$, and note that $y \in C_{y,H}$, $H \cap C_{y,H} \neq \phi$, and $C_{y,H}$ is the image of a subcontinuum of X (namely $\mathrm{cl}(\bigcup_{i=1}^{\infty} C_i)$). In addition, $C_{y,H}$ is not unique since it depends on the choice of x in $f^{-1}(y)$.

The following theorem, a corollary of [4, Theorem 2, p. 635], is needed.

Theorem A. *Let Y be a hereditarily decomposable θ_n-continuum. Then Y admits a monotone, upper-semicontinuous decomposition $\mathscr{D}$, the elements of which have void interior and which is unique and minimal with respect to the property that the quotient space $Y/\mathscr{D}$ is a graph.*

Each element of the decomposition is called a *tranche*, the *order* of a tranche is the order of the tranche considered as a point in the quotient space. If D_1 and D_2 are two tranches in $\mathscr{D}$, a connected subset C of $Y \backslash (D_1 \cup D_2)$ that is the union of tranches and has limit points in D_1 and D_2 is called a *segment* between D_1 and D_2, if all tranches in C are of order 2 (note that D_1 or D_2 may not be of order 2). Then D_1 and D_2 are called *end tranches* of the segment and the usual real number notation $[D_1, D_2]$, $[D_1, D_2)$, $(D_1, D_2]$, (D_1, D_2), specifies whether D_1 and D_2 is to be included in the segment.

Proof of the main theorem

We start with several lemmas that lead to a proof of the main theorem.

Lemma 1. *If Y is a continuum that contains no W-sets and for some integer $n \geq 3$ has no n-ods, then every proper, nondegenerate subcontinuum of Y is decomposable.*

Proof. Suppose Y contains a proper, indecomposable subcontinuum I, and let f be a map from a continuum X onto Y such that no subcontinuum of X maps onto I. For each natural number k, let P_k be the statement that Y contains mutually disjoint subcontinua, $H_1, \ldots, H_k$ such that for each i, $1 \leq i \leq k$, $H_i \backslash I \neq \phi$, $H_i \cap I \neq \phi$, and $I \backslash \bigcup_{i=1}^{k} H_i \neq \phi$. Choose points z, w in different composants of I and consider the continuum $C_{z,\{w\}}$. Since no subcontinuum of X maps onto I and $\{z, w\} \subset C_{z,\{w\}}$, $C_{z,\{w\}} \not\subset I$. There exists an integer i such that $C^i_{z,\{w\}} \not\subset I$. But $w \notin C^i_{z,\{w\}}$. Hence it follows that $H_1 = C^i_{z,\{w\}}$ and I are as required in P_1. Suppose P_k is true, and let $H_1, \dots, H_k$ be subcontinua as required in P_k. Assume for some i, $1 \le i \le k$, $H_i \cap I$ is not connected, and let $H_i \cap I = A \cup B$, a separation. There exist open sets U and V (open relative to H_i) with disjoint closures such that $A \subset U$ and $B \subset V$. By [2, Theorem 50, p. 18], there exist continua A_1, B_1 in H_i, such that $A_1 \subset \text{cl}(U)$, $B_1 \subset \text{cl}(V)$, $A_1 \cap A \ne \phi \ne A_1 \cap \text{bd}(U)$, and $B_1 \cap B \ne \phi \ne B_1 \cap \text{bd}(V)$. Then $\{H_j : 1 \le j \le k, j \ne i\} \cup \{A_1, B_1\}$ are $k+1$ subcontinua of Y as required in P_{k+1}. Assume that for each i, $1 \le i \le k$, $H_i \cap I$ is connected, let $H = \bigcup_{i=1}^k H_i$, and choose z in a different composant of I from the composants that contain $\{H_i \cap I : 1 \le i \le k\}$. Since no subcontinuum of X maps onto I and $C_{z,H}$ contains z and intersects H, $C_{z,H} \not\subset I$. There exists an integer i such that $C^i_{z,H} \not\subset I$. But $C^i_{z,H} \cap H = \phi$, so it follows that $C^i_{z,H}$ is a subcontinuum H_{k+1} that together with $H_1, \dots, H_k$ are the required continua in P_{k+1}. By induction, P_n is true. However, $I \cup (\bigcup_{i=1}^n H_i)$ is an n-od, a contradiction, so the lemma is true.

Lemma 2. *If Y is a continuum that contains no W-sets and, for some integer $n \ge 3$, has no n-ods, then every proper subcontinuum of Y is locally connected.*

Proof. By Lemma 1, each proper, nondegenerate subcontinuum W of Y is decomposable. By hypothesis, Y (and hence W) contains no n-ods, and W is therefore a θ_{n-1}-continuum. Theorem A guarantees that W admits a unique monotone, upper-semicontinuous decomposition $\mathscr{D}$ for which the tranches have void interior and for which $W/\mathscr{D}$ is a graph. To show local connectivity, it suffices to prove that each tranche in $\mathscr{D}$ is degenerate. Suppose T is a nondegenerate tranche in $\mathscr{D}$. Then there exists a segment $[T, T_1]$ and a nondegenerate subcontinuum T' of T such that $T' = T \cap \text{cl}((T, T_1])$. Let X be a continuum and f a map of X onto Y such that no subcontinuum of X maps onto T'. Suppose for each tranche T^* in $(T, T_1]$ there exists a continuum H^* in X such that $f(H^*) \cap (T, T^*) \ne \phi$ and $f(H^*) \cap (Y \backslash [T, T^*]) \ne \phi$, but $f(H^*) \cap (T \cup T^*) = \phi$. Then there exist $n+1$ tranches, $T_1, \dots, T_{n+1}$, and n subcontinua of X, $H_1, \dots, H_n$ such that $T < T_{n+1} < \cdots < T_1$, and for $1 \le i \le n$, $f(H_i) \cap (T_{i+1}, T_i) \ne \phi$, $f(H_i) \cap Y \backslash [T_{i+1}, T_i] \ne \phi$, but $f(H_i) \cap (T_i \cup T_{i+1}) = \phi$. Then $[T, T_1] \cup \bigcup_{i=1}^n f(H_i)$ contains an n-od, a contradiction.

Hence, there exists a tranche (assume without loss of generality that it is T_1) such that for each y in (T, T_1) and each i, $C^i_{y,K} \subset (T, T_1)$ where $K = T_1 \cup T$. Then $C_{y,K} \subset [T, T_1]$ and either $C_{y,K} \cap T_1 \ne \phi$ or $C_{y,K} \cap T' \ne \phi$. There exists a continuum H in X such that $f(H) \subset [T, T_1]$ and $f(H) \cap T' \ne \phi \ne f(H) \cap (T, T_1]$. To see this, note that it follows immediately if $C_{y,K} \cap T \ne \phi$ for some y in (T, T_1). So assume that $C_{y,K} \cap T_1 \ne \phi$ and $C_{y,K} \cap T' = \phi$ for each y in (T, T_1). Then by taking a suitable limit H of the continua in X that map onto the $C_{y,K}$, H is the required continuum. So $f|H$ is a map from

H into $[T, T_1]$ such that the image of H intersects $(T, T_1]$ and contains T', since $\mathrm{cl}(T, T_1)$ is irreducible between $T' = T \cap \mathrm{cl}(T, T_1)$ and $T_1 \cap \mathrm{cl}(T, T_1)$ [1, Lemma 1, p. 262].

Now for each y in $f(H)$ and each closed subset K of $f(H)\setminus\{y\}$, let us consider the sets $C_{y,K}$ relative to $f|H$. Fix a point z in T', and consider any other point y in T'. For each i, $C^i_{z,\{y\}}$ contains z but not y, so $C^i_{z,\{y\}} \subset T'$ [1, Lemma 1, p. 262]. Therefore, $C_{z,\{y\}} \subset T'$. Then if the same point x in $f^{-1}(z) \cap H$ is used to define $C_{z,\{y\}}$ for each y in $T'\setminus\{z\}$, then $T' = \{z\} \cup \{C_{z,\{y\}} : y \in T'\setminus\{z\}\}$ will be the image of a subcontinuum of H under f. But T' was assumed to not be the image of any subcontinuum in X under f. This contradiction establishes the lemma.

Lemma 3. *If Y is a continuum that contains no W-sets and, for some integer $n \geq 3$, has no n-ods, then Y is decomposable.*

Proof. Suppose Y is indecomposable. It follows from Lemma 2 that each composant is arcwise connected. If the order of a point in y in Y is defined as the maximum integer k such that there is a simple k-od in Y with hub y, then it follows that each composant contains at most a finite number of points whose order is different from 2. For, if not, it would be possible to construct a k-od for arbitrarily large k. Choose a composant C, and let K be a subcontinuum of C irreducible about the points in C of order different from 2 (if there are no points of order different from 2, let K be any singleton set). Let $\mathscr{D}$ be the monotone, upper-semicontinuous decomposition of Y whose only nondegenerate element is K (unless K is degenerate, in which case $\mathscr{D}$ is the set of singletons). Clearly $Y' = Y/\mathscr{D}$ is an indecomposable continuum. Denote K, as a point of Y', by k'. Since Y', as well is Y, contains no n-ods, there is a ray R, i.e., R is the continuous one-to-one image of $[0, \infty)$, in the composant of Y' containing k' whose only endpoint is k' and which is dense in Y'. Choose r in R and an open set U such that $r \in U$ and $k' \notin \mathrm{cl}(U)$. Every nondegenerate subcontinuum contained in $\mathrm{cl}(U) \cap R$ is an arc since all the points in C of order different from 2 are contained in K. Since R is dense in Y', in $\mathrm{cl}(U) \cap R$ there exist mutually disjoint arcs $A, A_1, A_2, \ldots$ such that $r \in A$, $\lim A_i = A$, and r is between k' and A_i on R for each i. For each i, let the endpoints of A_i be a_i and b_i and let the endpoints of A be p and q. Because A is not a W-set, there exist a continuum X and a map f from X onto Y such that no subcontinuum of X maps onto A. Since either p or q cuts the other from K in C, we can assume without loss of generality that p cuts q from K. For each j, either a_j or b_j cuts the other from q. Renaming if necessary, assume that a_j cuts b_j from q for each j. Let $H_j = \{p, b_j\}$ for each j, and consider C_{q,H_j} for a specific j. Since C^i_{q,H_j} for each i lies in the arc from p to b_j, C_{q,H_j} lies in the arc and contains either p or b_j. If, for some j, C_{q,H_j} contains p, then C_{q,H_j} contains A. Since an arc is in class W and there is a continuum in X that maps onto the arc C_{q,H_j}, there exists a continuum in X that maps onto A. But this is not possible, because no subcontinuum of X maps onto A.

So for each j, C_{q,H_j} contains b_j, and hence A_j, but not p. Hence there exists a continuum T_j in X such that $f(T_j) = A_j$. Choosing subsequences if necessary, let $T = \lim T_j$. Since $\lim A_j = A$, it follows that $f(T) = A$. Again, this is not possible, and the contradiction establishes the lemma.

The continuum Y of Lemmas 1, 2, and 3, is hereditarily decomposable and every proper subcontinuum is locally connected. So Y is a θ_{n-1}-continuum that is locally connected since all its tranches are degenerate. Therefore, Y is equivalent to its unique minimal monotone, upper-semicontinuous decomposition for which the quotient space is a graph. Hence Y is a graph.

We now prove the main theorem of the paper.

Theorem 1. *The continuum Y contains no W-sets and, for some integer $n \geq 3$, has no n-ods if and only if Y is a graph in which each point is contained in a simple closed curve.*

Proof. To prove the necessity, we will use the fact that Y is a graph. Suppose y is a point of order 2 that is not contained in a simple closed curve. Then y is a separating point of Y since Y is locally connected. Hence, there exists a segment A containing y such that A separates Y. Let $Y \backslash A = P \cup Q$, a separation, and denote by p and q the endpoints of A such that $\mathrm{cl}(P) \cap A = \{p\}$, $\mathrm{cl}(Q) \cap A = \{q\}$. Because A is not a W-set, there exist a continuum X and a map f from X onto Y such that no subcontinuum of X maps onto A. Let A_p be the set of all points x in $f^{-1}(A)$ such that the component of x in $f^{-1}(A)$ has a limit point in $f^{-1}(p)$. Define A_q analogously. Then $f^{-1}(A) = A_p \cup A_q$ and, because A is not a W-set, $\mathrm{cl}(A_p) \cap A_q = \phi = \mathrm{cl}(A_q) \cap A_p$; i.e., A_p and A_q are disjoint closed sets. Then $[f^{-1}(P) \cup A_p] \cup [f^{-1}(Q) \cup A_q]$ is a separation of X, a contradiction, and so every point of order 2 is contained in a simple closed curve. There are only a finite number of points in Y of order different from 2. If y is such a point, then there is a segment in Y with one endpoint y. Then the simple closed curve that contains one of the points of the segment contains y. This completes the proof of the necessity.

For the sufficiency, let Y be a graph such that each point is contained in a simple closed curve. A graph has only a finite number of points of order different from 2, and each point of order k different from 2 is the hub of a simple k-od (in fact, $k > 2$ since there are no endpoints in Y). It follows that Y contains no n-ods for some integer n sufficiently large.

Next, suppose that K is a proper nondegenerate subcontinuum of Y. Let $[a, b]$ be a segment that is properly contained in K, and let x_1, x_2, z, y_2, y_1 be points of (a, b) in the order from a to b. Choose r in $Y \backslash K$. Define a function f from Y onto Y which is the identity on $Y \backslash [a, b]$, and maps the following arcs homeomorphically: ax_1 onto an arc ar not containing z, by_1

onto an arc br not containing z, x_1x_2 onto rax_1x_2z, y_1y_2 onto rby_1y_2z, x_2z onto zx_2x_1ar, and y_2z onto zy_2y_1br. No subcontinuum of Y maps onto K, and thus the proof is complete.

ADDITIONAL FACTS

The next theorem discusses the nature of the graphs yielded by Theorem 1.

Theorem 2. *Let Y be a graph with the property that every point is contained in a simple closed curve. If n is an integer, $n \geq 2$, such that Y contains an n-od but no $(n+1)$-od, then n is even and Y is the union of $n/2$ simple closed curves such that no one curve is contained in the union of the others.*

Proof. Let K be an n-od in Y with hub H. Then K contains all the junction points of Y.

To see this, suppose that y is a junction point of Y that is not in K, and let A be an arc in Y from K to y. There exist segments B and C such that $B \cap K = \phi$, $C \cap K = \phi$, and $A \cap B = A \cap C = B \cap C = \{y\}$. But $K \cup A \cup B \cup C$ contains an $(n+1)$-od, a contradiction. In fact, the hub H contains all the junction points. To see this, assume that the junction point $y \notin H$. Then y lies in Q, one of the n connected sets that comprise $K \backslash H$. Let A be an arc in $\mathrm{cl}(Q)$ from y to H, and let B and C be segments such that $B \cap H = \phi$, $C \cap H = \phi$, and $A \cap B = A \cap C = B \cap C = \{y\}$. Then $(K \backslash Q) \cup A \cup B \cup C$ is an $(n+1)$-od with hub $H \cup A$, a contradiction.

The n-od K contains no simple closed curve. Assume that it does. Then there exists an open segment A in K such that $K \backslash A$ is connected and is an n-od. Then K is an $(n+1)$-od, a contradiction.

Since K is an n-od, let $K = H \cup \bigcup_{i=1}^{n} Q_i$ where the Q_i's are mutually disjoint connected sets. Each component of $Y \backslash K$ is an open segment with endpoints in different Q_i's. Also, different components of $Y \backslash K$ cannot have endpoints in the same Q_i or else Y would contain an $(n+1)$-od. So n is even and $Y \backslash K$ has $n/2$ components. Each component of $Y \backslash K$ is contained in a simple closed curve; label these simple closed curves $S_1, \ldots, S_{n/2}$. Then $\bigcup_{i=1}^{n/2} S_i$ has the property that no S_i is contained in the union of the others. Since K contains no simple closed curve, $Y = \bigcup_{i=1}^{n/2} S_i$, and the theorem is proved.

Note that K in Theorem 2 is a finite tree, and if $\mathscr{D}$ is the monotone, upper-semicontinuous decomposition whose only nondegenerate element is K, then $Y/\mathscr{D}$ is a generalized figure-8 curve with $n/2$ loops.

Theorems 1 and 2 also yield the following corollary.

Corollary 1. *If Y is an atriodic continuum that contains no W-sets, then Y is a simple closed curve and conversely.*

Theorem 1 can be slightly strengthened as follows. If we define a proper, nondegenerate subcontinuum K of Y to be a *W'-set* in Y if for every map

f of Y onto Y, some subcontinuum of Y is mapped onto K, then a W-set is clearly a W'-set. The following theorem is a slightly stronger version of Theorem 1.

Theorem 3. *The continuum Y contains no W'-sets and for some integer $n \geq 3$ has no n-ods if and only if Y is a graph in which each point is contained in a simple closed curve.*

Proof. Clearly if Y contains no W'-sets, then Y contains no W-sets, so the necessity follows from Theorem 1. Since the function f defined in the proof of sufficiency in Theorem 1 is a map from Y onto Y, the sufficiency also follows from Theorem 1.

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