HIGHER-DIMENSIONAL SHIFT EQUIVALENCE
AND STRONG SHIFT EQUIVALENCE
ARE THE SAME OVER THE INTEGERS

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Abstract. Let $RS(\Lambda)$ and $S(\Lambda)$ denote, respectively, the spaces of strong shift equivalences and shift equivalences over a subset $\Lambda$ of a ring which is closed under addition and multiplication. For example, let $\Lambda$ be the integers $\mathbb{Z}$ or the nonnegative integers $\mathbb{Z}^+$. For any principal ideal domain $\Lambda$, we prove that the continuous map $RS(\Lambda) \to S(\Lambda)$ is a homotopy equivalence. The methods also show that any inert automorphism, i.e., an element in the kernel of $\pi_1(RS(\mathbb{Z}^+), A) \to \pi_1(S(\mathbb{Z}^+), A)$ can be represented by a closed loop in $RS(\mathbb{Z}^+)$ which in $RS(\mathbb{Z})$ is spanned by a triangulated 2-disc supporting a positive 1-cocycle. These cocycles are used in work of Kim-Roush that leads to a counterexample to Williams' lifting problem for automorphisms of finite subsystems of subshifts of finite type.

1. Introduction

R. F. Williams [Wi] introduced shift equivalence SE and strong shift equivalence SSE over the non-negative integers $\mathbb{Z}^+$ in connection with the classification of subshifts of finite type $(X_A, \sigma_A)$ up to topological conjugacy, and it remains an open question whether SE and SSE are the same over the non-negative integers. He did prove that SE and SSE are the same over the ring of integers $\mathbb{Z}$. Study of the automorphism group $\text{Aut}(\sigma_A)$ of $(X_A, \sigma_A)$ led in [W1, W2, W3] to the introduction of the CW-complexes $RS(\Lambda)$ of strong shift equivalences and $S(\Lambda)$ of shift equivalences over a subset $\Lambda$ of a ring which is closed under addition and multiplication. It follows from the definitions of these spaces that the set $\pi_0(RS(\Lambda))$ of path components of $RS(\Lambda)$ is just the set of SSE classes over $\Lambda$ and that $\pi_0(S(\Lambda))$ is the set of SE classes over $\Lambda$. A step in [W3] for obtaining eventual finite-order generation for inert automorphisms is the isomorphism [W3, 1.7] between the group of automorphisms $\text{Aut}(\sigma_A)$...
modulo the simple ones and $\pi_1(RS(Z^+), A)$. As known to Boyle, Fiebig, and Krieger for several years, results of the type obtained by Kim and Roush in their paper [KR] would lead to examples of a shift commuting automorphism of the periodic points of order 6 of the full 2-shift which is not the restriction of a shift commuting automorphism of the 2-shift itself. A key insight in [KR] is how to convert certain statements over $Z$ to other appropriate ones over $Z^+$. In particular, a fact they use is that $\pi_1(RS(Z), A) \rightarrow \pi_1(S(Z), A)$ is a monomorphism. Kim and Roush also introduce the concept of a "positive 1-cocycle" and need to know that any inert automorphism, i.e., an element in the kernel of $\pi_1(RS(Z^+), A) \rightarrow \pi_1(S(Z^+), A)$, can be represented by a closed loop in $RS(Z^+)$, which in $RS(Z)$ is null-homotopic by a triangulated 2-disc supporting a positive 1-cocycle. The specific construction of such cocycles given in (2.7), (2.10), and (2.16) of this paper is used in [KR].

(1.1) Theorem. Let $\Lambda$ be a principal ideal domain. Then $RS(\Lambda) \rightarrow S(\Lambda)$ is a homotopy equivalence and $\pi_n(RS(\Lambda), A) = \pi_n(S(\Lambda), A) = 0$ for $n \geq 2$.

Consider Krieger's dimension group constructed over $\Lambda$ from the stationary system $A: \Lambda^m \rightarrow \Lambda^m$. (See [BLR] and [Wl].) Let $\text{Aut}(s_{A/\Lambda})$ denote the group of automorphisms of this dimension group which commute with $s_{A/\Lambda}$, but which do not necessarily preserve the order structure. In fact, there generally is not an order structure on the dimension group unless $\Lambda$ is ordered anyway.

(1.2) Proposition. $\pi_1(S(\Lambda), A) = \text{Aut}(s_{A/\Lambda})$.

The proof of (1.2) is entirely similarly to [Wl, 4.18]. Simply ignore any considerations of the order structure.

2. Proof of the main theorem (1.1) for $\pi_0$ and $\pi_1$

Step 1. $\pi_0(RS(\Lambda), A) \rightarrow \pi_0(S(\Lambda), A)$ is a bijection of sets.

From the definitions we know immediately that this map is surjective. Williams also showed that this map is injective (at least for $\Lambda = Z$). For completeness, we include a proof by showing

(2.1) $\pi_1(S(\Lambda), RS(\Lambda)) = 0$

In other words, any path from $A$ to $B$ in $S(\Lambda)$ can be deformed back to a path from $A$ to $B$ in $RS(\Lambda)$ keeping endpoints fixed. Notation that will be used in the remainder of the paper is introduced in the proof of (2.1).

Any path from $A$ to $B$ to $S(\Lambda)$ is the concatenation of elementary shift equivalences $R: P \rightarrow Q$ or their inverses. So the proof reduces to verifying (2.1) for $R: P \rightarrow Q$. We now show how to reduce the argument further to the case where each of $P$ and $Q$ are monomorphisms and therefore are isomorphisms when tensored by $F(\Lambda)$, the field of fractions of $\Lambda$. This implies $R: P \rightarrow Q$ is also a monomorphism and an isomorphism when tensored with $F(\Lambda)$. By definition, each vertex $M$ of $RS(\Lambda)$ and of $S(\Lambda)$ is a square $m \times m$-matrix over $\Lambda$ where $m$ can vary from vertex to vertex. View $M$ as an endomorphism of
the free $A$-module $V$ of rank $m$ equipped with the standard basis with respect to which the endomorphism $M$ has the original matrix representation $M$. We write this as $\{V, M\}$. More generally, we let $\{V, M\}$ denote a free $A$-module of finite rank equipped with a basis and an endomorphism $M$. If $M$ comes from an $m \times m$-matrix, we always take the basis to be the standard one. Now consider the SSE

$$\text{(2.2)}$$

$$(\pi, M): \{V, M\} \to \{V/\ker M, M\}$$

where $\pi: V \to V/\ker M$ is the projection and both $M: V/\ker M \to V$ and $M: V/\ker M \to V/\ker M$ are the homomorphisms induced by $M$. While $V$ comes with a basis, $V/\ker M$ does not have a canonical one. So we will choose a basis for $V/\ker M$ in such a way that if $\ker M = 0$, then the basis for $V/\ker M = V$ remains the same. This makes (2.2) a SSE in the category of matrices over $\Lambda$.

The subgroup of those elements in $V$ which are killed by some power of $M$ is free and finitely generated because $\Lambda$ is a PID and $V$ is free of finite rank over $\Lambda$. So the construction (2.2) may be repeated a finite number of times to produce a chain of integral SSEs to a monomorphism. Moreover, starting with the shift equivalence $R: \{V, P\} \to \{W, Q\}$ over $\Lambda$, there is the following commutative diagram in $\mathcal{S}(\Lambda)$:

$$\text{(2.3)}$$

$$\begin{array}{c}
\{V, P\} \xrightarrow{R} \{W, Q\} \\
\downarrow (\pi, P) \quad \downarrow R \pi \quad \downarrow (\pi, Q) \\
\{V/\ker P, P\} \quad \downarrow R \quad \{W/\ker Q, Q\}.
\end{array}$$

Continuing this procedure a finite number of times deforms $R: \{V, P\} \to \{W, Q\}$ to a monomorphism as required.

Assume that $R: \{V, P\} \to \{W, Q\}$ is a SE where $P$, $Q$, and $R$ are monomorphisms. Choose $S: \{W, Q\} \to \{V, P\}$ and $k > 0$ such that $RS = P^k$ and $SR = Q^k$. Let $VR$ be the image of $R$ in $W$. Then we have the following diagram.

$$\text{(2.4)}$$

$$\begin{array}{c}
\{V, P\} \xrightarrow{R} \{W, Q\} \\
\downarrow (R, SP^{-(k-1)}) \\
\{VR, Q\} \xrightarrow{(I, Q)} \{VR + WQ^{k-1}, Q\} \\
\downarrow (I, Q) \\
\{VR + WQ^{k-2}, Q\} \xrightarrow{(I, Q)} \ldots \xrightarrow{(I, Q)} \{VR + WQ, Q\}.
\end{array}$$
As above, we arbitrarily choose bases for the $VR + WQ^j$ to get a diagram in $S(\Lambda)$. This completes the proof of Step 1.

**Step 2.** $\pi_1(RS(\Lambda), A) \rightarrow \pi_1(S(\Lambda), A)$ is an isomorphism.

From (2.1), we know this is surjective. Thus, we must prove it is injective. This will be done in a way which produces the “positive 1-cocycles” used by Kim-Roush in [KR].

Consider an elementary SSE $(R, S): \{V, P\} \rightarrow \{W, Q\}$. We have the diagram

\[ \begin{array}{ccc}
\{V, P\} & \xrightarrow{(\pi, P)} & \{V/\ker R, P\} \\
\{V/\ker P, P\} & \xrightarrow{(\pi, P^*)} & \{W/\ker Q, Q\} \\
\end{array} \]

in $RS(\Lambda)$. Recall from [W2, W3] that we often let $\gamma(R, S)$ denote the path in $RS(\Lambda)$ corresponding to the elementary strong shift equivalence $(R, S): P \rightarrow Q$. Moreover, we have the identities

\[ \gamma(R, S)\gamma(S, R) = \gamma(P, 1), \]
\[ \gamma(S, R)\gamma(R, S) = \gamma(Q, 1), \]

and

\[ \gamma(1, P) = \gamma(1, Q) = 1. \]

Using these, we can represent any element of $\pi_1(RS(\Lambda), A)$ as a loop

\[ V = V_0 \rightarrow V_1 \rightarrow \cdots \rightarrow V_n \leftarrow \cdots \leftarrow V_{2n-1} \leftarrow V_{2n} = V, \]

where the number of forward and backward arrows is the same. Each free $\Lambda$-module $V_i$ comes equipped with an endomorphism $A_i$ and a basis giving a matrix representation for $A_i$. For example, if $\Lambda = \mathbb{Z}^+$ and the loop represents an element of $\mathrm{Aut}(\sigma_A)$, then the matrices are non-negative. For simplicity we have omitted the notation for the $A_i$, as well as for the $R$ and $S$ matrices giving the SSEs between the vertices in the path. Applying the procedure in (2.5) a finite number of times produces a homotopy of the loop (2.6) like the following ($n = 2$):

\[ \begin{array}{ccc}
2 & 1 & 2 \\
2 & 1 & 2 \\
1 & 1 & 2 \\
1 & 1 & 2 \\
1 & 1 & 2 \\
2 & 1 & 2 \\
2 & 1 & 2 \\
\end{array} \]
where the vertex endomorphisms and the homomorphisms in the elementary SSEs are all monomorphisms along the bottom loop. Clearly, the construction applies to all path lengths $2n$. The right and left vertical paths are identified to obtain a homotopy of closed loops. The numbers along each edge give a specific positive 1-cocycle on the homotopy.

This notion was introduced by Kim and Roush. It consists of a function $f$ from the oriented edges of the homotopy to the positive integers satisfying the cocycle condition $f(z) = f(x) + f(y)$ whenever $x$, $y$, and $z$ form the boundary of an oriented triangle as in the diagram

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Now we assume the vertex $A$ matrices and the edge $(R, S)$ matrix pairs in the loop (2.6) are monomorphisms and we must show that when (2.6) is inert, it is possible to deform it to a point by a homotopy supporting a positive 1-cocycle fitting together with the one in (2.7). For the remainder of this section all endomorphisms and homomorphisms will be assumed to be monomorphisms.

Fix a monomorphism $A : V \to V$ of the free $\Lambda$-module $V$. As noted earlier, let $F(\Lambda)$ denote the field of fractions of $\Lambda$. Let $\text{Lat}(A)$ denote the set of lattices in $V \otimes F(\Lambda)$ which are invariant under $A$. Make $\text{Lat}(A)$ into a simplicial complex by letting an $n$-simplex be an $(n+1)$-tuple $[L_0, \ldots, L_n]$ where each $L_i$ is a lattice in $V \otimes F(\Lambda)$ such that $L_i \subset L_j$ and $A(L_i) \subset L_i$ whenever $i < j$.

(2.8) Proposition. Each connected component of $\text{Lat}(A)$ is contractible.

First, we prove that each component is simply connected. We use the notation $K \to L$ to mean $[K, L]$ is an edge in $\text{Lat}(A)$. Similarly for triangles. We give the argument for the connected component containing $V$. The argument for other components is the same. So start with a loop

(2.9) $V = L_0 \to L_1 \to \cdots \to L_n \leftrightarrow \cdots \leftrightarrow L_{2n-1} \leftrightarrow L_{2n} = V$

in $\text{Lat}(A)$. The well-known formula for the product of a matrix and its classical adjoint implies that the intersection of two lattices in $V \otimes F(\Lambda)$ is again a lattice. Apply this to produce the null-homotopy of (2.9) in $\text{Lat}(A)$ when $n = 2$ as in diagram (2.10) below.

The numbers along the edges give a positive 1-cocycle. The construction clearly generalizes to any path length $2n$. The two 45-degree paths arising from the lowest vertex are identified to obtain a homotopy of closed loops. The diagram in (2.10) resembles the one in the proof of [BFK, 2.12].
To finish the proof of (2.8), we must show that the higher homotopy groups of each component of \( \text{Lat}(A) \) vanish. By the Whitehead Theorem [Sp], it suffices to show that the higher homology groups are zero, because we have just shown that the components are simply connected. The proof for this is essentially a verbatim copy of the proof for Step III in [W1, 2.12]. The notation \( U \to V \) for Markov partitions is replaced by the notation \( K \to L \) for lattices.

Finally, we can finish the proof of Step 2 by fitting (2.7) and (2.10) together via (2.16) below. Represent an inert element \( \alpha \) of \( \pi_1(\text{RS}(\Lambda), A) \) by a loop

\[
\alpha = \prod_{i=1}^{2n} \gamma(R_i, S_i)^{\varepsilon_i}
\]

where \((R_i, S_i): (V_{i-1}, A_{i-1}) \to (V_i, A_i)\) and \( \varepsilon_i = +1 \) for \( 1 \leq i \leq n \), and \((R_i, S_i): (V_i, A_i) \to (V_{i-1}, A_{i-1})\) and \( \varepsilon_i = -1 \) for \( n+1 \leq i \leq 2n \). For \( 1 \leq i \leq 2n \), let \( B_i \) denote the isomorphism from \( V \otimes F(\Lambda) = V_0 \otimes F(\Lambda) \) to \( V_i \otimes F(\Lambda) \) given by the formula

\[
B_i = \prod_{p=1}^{i} (R_p \otimes 1)^{\varepsilon_p}
\]

Let \( B_0 = I \). Observe that since \( \alpha \) is inert, we also have \( B_{2n} = I \). Moreover, we have \( A = B_i B_i^{-1} \) for each \( i \). Let \( W_i \) be the lattice in \( V \otimes F(\Lambda) \) defined by the equation

\[
W_i B_i = V_i.
\]

Then we have the following two diagrams in \( \text{RS}(\Lambda) \) for the two cases \( \varepsilon_i = +1 \) and \( \varepsilon_i = -1 \), respectively.
Diagrams (2.14) and (2.15) piece together to form the homotopy in (2.16) below from the bottom line in (2.7) to the top line in (2.10). Again, we take $2n = 4$ for simplicity. The construction clearly generalizes to all lengths $2n$.

\begin{equation}
\begin{array}{c}
V_{i-1} \\
\downarrow (B_{i-1}^{-1}, B_{i-1} A_{i-1}) \\
W_{i-1} \\
\end{array} 
\begin{array}{c}
V_i \\
\downarrow (R_i, S_i) \\
W_i \\
\end{array}
\end{equation}

The positive 1-cocycle compatible with those in (2.7) and (2.10) is marked on the diagram. This completes the proof of Step 2.

3. Proof of theorem (1.1) for $\pi_n$ when $n \geq 2$

Let $MRS(\Lambda)$ and $MS(\Lambda)$ denote the subspaces of $RS(\Lambda)$ and $S(\Lambda)$, respectively, where each SSE or SE is a monomorphism. Using generalizations of (2.3) and (2.5), we first show

(3.1) Theorem. Assume $\Lambda$ is a principal ideal domain. Then the inclusions $MS(\Lambda) \subset S(\Lambda)$ and $MRS(\Lambda) \subset RS(\Lambda)$ are homotopy equivalences.

Proof that $MS(\Lambda) \subset S(\Lambda)$ is a homotopy equivalence.

Let $\Delta = [V_0, \ldots, V_n]$ be an $n$-simplex in $S(\Lambda)$, where each $V_i$ is a free, based $\Lambda$-module of finite rank with an endomorphism $A_i$ such that whenever $i < j$, we are also given a shift equivalence $R_{ij}: V_i \to V_j$ between $A_i$ and $A_j$ satisfying $R_{ij} R_{jk} = R_{ik}$ for $i < j < k$. As in (2.2) and (2.3), let $\Delta' = [V_0/\ker A_0, \ldots, V_n/\ker A_n]$ with the corresponding vertex endomorphisms $A_i$ and edge shift equivalences $R_{ij}$ induced from those in $\Delta$. Let $I$ denote the unit interval. Then, as in [S], we can triangulate $\Delta \times I$ as follows: The vertices consist of the vertices $V_i$ of $\Delta$ together with the vertices $V_i/\ker A_i$ of $\Delta'$. The directed edges are of the form $R_{ij}: V_i \to V_j$, $R_{ij}: V_i/\ker A_i \to V_j/\ker A_j$, $R_{ij}: V_i \to V_j/\ker A_j$ when $i < j$, and $V_i \to V_i/\ker A_i$. These satisfy the triangle relation $R_{ij} R_{jk} = R_{ik}$ for $i < j < k$, and the procedure gives

(3.2) a deformation of $\Delta$ to $\Delta'$ compatible with the face and degeneracy maps in the complex $S(\Lambda)$.

Consider a map of a pair $(X, Y)$ of finite CW-complexes into the pair $(S(\Lambda), MS(\Lambda))$. Deform this to a cellular map so that the image of $X$ is contained in a finite subcomplex of $S(\Lambda)$. Since $\Lambda$ is a principal ideal domain, we can use the deformation (3.2) a finite number of times to deform the map on $X$ down into $MS(\Lambda)$ keeping it fixed on $Y$. 

HIGHER-DIMENSIONAL SHIFT EQUIVALENCE AND STRONG SHIFT EQUIVALENCE 533
Proof that $MRS(\Lambda) \subset RS(\Lambda)$ is a homotopy equivalence.

This proof proceeds exactly like the argument for the previous proof except that, just as (2.5) is more complicated than (2.3), so is the triangulation of $\Delta \times I$.

Let $\Delta = [V_0, \ldots, V_n]$ be an $n$-simplex in $RS(\Lambda)$ where each $V_i$ is a free, based $\Lambda$-module of finite rank equipped with an endomorphism $A_i$ and such that whenever $i < j$, we are also given a strong shift equivalence $(R_{ij}, S_{ji})$: $V_i \to V_j$ between $A_i$ and $A_j$ satisfying the Triangle Identities in [W3] for $i < j < k$. As in (2.2) and (2.5), let $\Delta' = [V_0/\ker A_0, \ldots, V_n/\ker A_n]$ with the corresponding vertex endomorphisms $A_i$ and edge strong shift equivalences $(R_{ij}, S_{ji})$ induced from those in $\Delta$. We will define a triangulation of $\Delta \times I$ which will be isomorphic to the cone from an interior “center” vertex $C$ of $\Delta \times I$ to an inductively defined triangulation of $\partial(\Delta \times I)$. For any simplex $\Delta$, let $C = V_0/\ker R_{01}$ equipped with the induced endomorphism $A_0$. The vertices of $\Delta \times I$ will be of four types:

(a) the $V_i$ of $\Delta$,
(b) the $V_i/\ker A_i$ of $\Delta'$,
(c) the centers $C_i$ of the faces $(\partial_i \Delta) \times I$ for $0 \leq i \leq n$, and
(d) the center $C$ of $\Delta \times I$.

From the definition of the centers, we see that $C_i$ is given by

$$C_i = \begin{cases} 
V_1/\ker R_{12}, & \text{for } i = 0 \\
V_0/\ker R_{02}, & \text{for } i = 1 \\
V_0/\ker R_{01}, & \text{for } i \geq 2 
\end{cases}$$

The edges are

- (ea) $(R_{ij}, S_{ji})$: $V_i \to V_j$ in $\Delta$,
- (eb) $(R_{ij}, S_{ji})$: $V_i/\ker A_i \to V_j/\ker A_j$ in $\Delta'$,
- (ec) the edges in $(\partial_i \Delta) \times I$, and
- (ed) a directed edge connecting $C$ and $V$ for each vertex $V$ of the three types (a), (b), and (c).

More specifically, in the situation (ed) we have the following cases.

If $V = V_0$ in $\Delta$, then $V \to C$ is

$$(\pi, A_0): V_0 \to V_0/\ker R_{01}.$$ 

If $V = V_i$ in $\Delta$ for $i \geq 1$, then $C \to V$ is

$$(R_{0i}, S_{i0})\pi: V_0/\ker R_{01} \to V_i.$$ 

If $V = V_i/\ker A_i$ in $\Delta'$ for $i \geq 0$, then $C \to V$ is

$$(\pi, A_0): V_0/\ker R_{01} \to V_0/\ker A_0$$

for $i = 0$, and

$$(R_{0i}, S_{i0})\pi: V_0/\ker R_{01} \to V_i/\ker A_i$$

for $i \geq 1$. 

V is of Type (c):
If $V = C_i$, then $C \to V$ is

$$(R_{0i}, S_{10}): V_0/\ker R_{01} \to V_1/\ker R_{12}, \text{ for } i = 0,$$

$$(\pi, A_0): V_0/\ker R_{01} \to V_0/\ker R_{02}, \text{ for } i = 1, \text{ and}$$

$$(I, A_0): V_0/\ker R_{01} \to V_0/\ker R_{01}, \text{ for } i \geq 2.$$  

The Triangle Identities are satisfied by any triple of edges which could possibly form a triangle, and hence we get a triangulation of $\Delta \times I$ in $RS(A)$. This gives

$$\text{(3.3)} \quad \text{a deformation of } \Delta \text{ to } \Delta' \text{ compatible with the face}$$

and degeneracy maps in the complex $RS(A)$.

Now we will proceed in the same manner to complete the proof of (3.1).

Proof that $\pi_n(MRS(A), A) = 0$ when $n \geq 2$.
This is very similar to §4 of [W2]. Let $MRS(A)_A$ denote the component of $MRS(A)$ containing $A$. The universal cover $\widetilde{MRS(A)}_A$ of $MRS(A)_A$ is the realization of the following simplicial set. The $n$-simplices are pairs $(\gamma, \Delta)$ where

$$\Delta \text{ is an } n\text{-simplex of } \widetilde{MRS}(A) \text{ given by the data } [V_0, \ldots, V_n]$$

and $(R_{ij}, S_{ji}): V_i \to V_j$, and $\gamma$ is a homotopy class of paths from $V$ to $V_0$.

The $i$th face operator acts on $\Delta$ just as it does in $MRS(A)_A$. For $1 \leq i \leq n$, it leaves $\gamma$ unchanged, and for $i = 0$, it changes $\gamma$ to $\gamma \ast \gamma(R_{01}, S_{10})$. The covering map

$$\widetilde{MRS}(A)_A \to MRS(A)_A$$

is induced by the map of simplicial sets taking $(\gamma, \Delta)$ to $\Delta$.

Let $\gamma$ be a path from $V$ to $V'$ in $MRS(A)_A$ written in the form

$$\gamma = \prod_{i=1}^{n} \gamma(R_i, S_i)^{\varepsilon_i},$$

where $(R_i, S_i): \{V_{i-1}, A_{i-1}\} \to \{V_i, A_i\}$ when $\varepsilon_i = +1$ and $(R_i, S_i): \{V_i, A_i\} \to \{V_{i-1}, A_{i-1}\}$ when $\varepsilon_i = -1$. Let $\Theta(\gamma)$ denote the isomorphism from $V \otimes F(A)$ to $V' \otimes F(A)$ given by the formula

$$\Theta(\gamma) = \prod_{i=1}^{n} (R_i \otimes 1)^{\varepsilon_i}.$$  

For each $n$-simplex $(\gamma, \Delta)$ of $\widetilde{MRS}(A)_A$, let $L_i$ be the lattice in $V \otimes F(A)$ that is the image of $V_i$ under $\Theta(\gamma \ast \gamma(R_{0i}, S_{10}))^{-1}$. Then $[L_0, \ldots, L_n]$ is an $n$-simplex in $\text{Lat}(A)$, and the correspondence taking $(\gamma, \Delta)$ to $[L_0, \ldots, L_n]$ is a map of simplicial sets. This gives a continuous map

$$\text{\widetilde{MRS}(A)}_A \to \text{Lat}(A).$$
Let $K$ and $L$ be $A$-invariant lattices in $V \otimes F(\Lambda)$ such that $A(L) \subset K$. This gives rise to a path $(I, A): K \rightarrow L$ in $MRS(\Lambda)_A$, and the correspondence preserves the Triangle Identities. Therefore, there is a continuous map

$$\text{Lat}(A) \rightarrow MRS(\Lambda)_A.$$ 

Moreover, it follows from [S] and from conjugation squares like (2.14) and (2.15) that there is a homotopy commutative diagram

$$\begin{array}{ccc}
\text{MRS}(\Lambda)_A & \rightarrow & \text{Lat}(A) \\
\downarrow & & \downarrow \\
MRS(\Lambda)_A & &
\end{array}$$

Since the components of $\text{Lat}(A)$ are contractible, we see that the homomorphism from $\pi_n(\text{MRS}(\Lambda)_A, A)$ to $\pi_n(MRS(\Lambda)_A, A)$ is zero. On the other hand, it is an isomorphism for $n \geq 2$, because $MRS(\Lambda)_A$ is the universal cover.

**Proof that** $\pi_n(MS(\Lambda), A) = 0$ **when** $n \geq 2$.

This proof follows just like the argument for $MRS(\Lambda)$, except that the definition of $\text{Lat}(A)$ is changed slightly. Namely, an $n$-simplex is an $(n + 1)$-tuple $[L_0, \ldots, L_n]$ where each $L_i$ is an $A$-invariant lattice in $V \otimes F(\Lambda)$ such that $L_i \subset L_j$, and there is a positive integer $k$ so that $A^k(L_j) \subset L_i$ whenever $i < j$.

**References**


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