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LIFTING HOMEOMORPHISMS OF A COMPLEX SURFACE TO ITS NORMALIZATION

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ABSTRACT. It is proved that any homeomorphism of a complex analytic surface lifts to a homeomorphism of the normalization.

The purpose of the present note is to provide a proof of the following result which was mentioned in the paper [3] of Steenbrink and Stevens without proof.

Theorem. *Any homeomorphism of a complex analytic surface lifts to a homeomorphism of the normalization.*

Before giving the proof of this theorem we recall the basic notations and results which are needed. By a complex analytic surface we mean a reduced, second countable, complex space of pure dimension 2. If $(X, \mathscr{O}_X)$ is a complex space with structure sheaf $\mathscr{O}_X$ and $a \in X$, then (X, a) will denote the germ of X at a. A theorem of Mumford [2] states that if the germ (X, a) is homeomorphic to the germ $(\mathbf{C}^2, 0)$, then a is a nonsingular point of X.

If X is a complex surface, then $S(X)$ will denote the singular subspace of X with the reduced structure. We say that a point $a \in S(X)$ is a *topological multicross* if the germ (X, a) can be written as a union of surface germs (X_i, a)

$$(X, a) = (X_1, a) \cup \cdots \cup (X_r, a)$$

where

(1) (X_i, a) is analytically isomorphic to $(\mathbf{C}^2, 0)$ for $1 \leq i \leq r$.
(2) $(X_i, a) \cap (X_j, a) = (S(X), a)$ for all $i \neq j$.
(3) $(S(X), a)$ is homeomorphic to $(\mathbf{C}, 0)$

A point $a \in S(X)$ is said to be an analytic multicross, or just a multicross, if the germ (X, a) satisfies conditions (1), (2), and the following condition $(3')$, instead of (3).

$(3')$ $(S(X), a)$ is analytically isomorphic to the germ $(\mathbf{C}, 0)$.

The multicross singularities of complex surfaces were introduced in [1] in order to provide a structure theorem for weakly normal surface singularities.

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Recall that a complex space is weakly normal if the sheaf $\mathscr{O}_X^c$ of germs of c-holomorphic functions on X is equal to the sheaf $\mathscr{O}_X$ of germs of holomorphic functions on X.

We now introduce notation for certain subsets of X: X_{reg} denotes the set of all regular (smooth) points of X, X_{tm} denotes the set of all topological multicross points of X, and X_{am} denotes the set of all analytic multicross points of X. Then we let $X_s = X \backslash (X_{\text{reg}} \cup X_{tm})$ and $X_u = X \backslash (X_{\text{reg}} \cup X_{am})$. Thus X can be written as a disjoint union in two different ways: $X = X_{\text{reg}} \cup X_{tm} \cup X_s = X_{\text{reg}} \cup X_{am} \cup X_u$. Since $X_{am} \subseteq X_{tm}$, it follows that $X_s \subseteq X_u$. If X is a weakly normal complex surface, then the Oka theorem for weakly normal singularities (Theorem 2.4 of [1]) shows that X_u (and hence $X_s \subseteq X_u$) is a discrete subset of X.

Lemma. *Let X be a complex surface and let $h: X \to X$ be a homeomorphism. Then $h(X_{\text{reg}}) = X_{\text{reg}}$, $h(X_{tm}) = X_{tm}$, and $h(X_s) = X_s$.*

Proof. If a is a regular point of X, then $h(a)$ is a regular point of X by Mumford's theorem. Thus $h(X_{\text{reg}}) = X_{\text{reg}}$. Suppose $a \in X_{tm}$. Then $(X, a) = (X_1, a) \cup \cdots \cup (X_{r(a)}, a)$ where (X_i, a) is analytically isomorphic to $(\mathbf{C}^2, 0)$ and $(X_i, a) \cap (X_j, a) = (S(X), a)$ for all $i \neq j$ and with $(S(X), a)$ homeomorphic to $(\mathbf{C}, 0)$. Now $h: (X, a) \to (X, h(a))$ is a homeomorphism of germs and $h(S(X)) = S(X)$ so that $h: (X \backslash S, a) \to (X \backslash S, h(a))$ is also a homeomorphism of germs, where $S(X) = S$. Therefore, $(X \backslash S, h(a))$ has the same number of connected components as $(X \backslash S, a)$, so that $(X, h(a))$ can be written as a union $(X_1', h(a)) \cup \cdots \cup (X_{r(a)}', h(a))$ and $h: (X, a) \to (X, h(a))$ decomposes (after reindexing, if necessary) into a union of homeomorphisms $h_i: (X_i, a) \to (X_i', h(a))$ with the property that $h_i|_{X_i \cap X_j = S(X)} = h_j|_{X_i \cap X_j = S(X)}$. By Mumford's theorem, $(X_i', h(a))$ is nonsingular for all i, and $(S(X), h(a))$ is homeomorphic to $(S(X), a)$, which is homeomorphic to $(\mathbf{C}, 0)$. Therefore, $h(a) \in X_{tm}$. Finally, $h(X_s) = X_s$ since X is a disjoint union of X_{reg}, X_{tm}, and X_s.

We now proceed to the proof of the theorem. Thus let X be a complex analytic surface, let $\eta: \tilde{X} \to X$ be the normalization of X, and let $\tilde{X}_{\text{reg}} = \eta^{-1}(X_{\text{reg}})$, $\tilde{X}_{tm} = \eta^{-1}(X_{tm})$, and $\tilde{X}_s = \eta^{-1}(X_s)$. Assume that $h: X \to X$ is a homeomorphism. Then we claim that there is a homeomorphism $\tilde{h}: \tilde{X} \to \tilde{X}$ such that the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{h}} & \tilde{X} \\ \downarrow \eta & & \downarrow \eta \\ X & \xrightarrow{h} & X \end{array}$$

is commutative. The proof proceeds by constructing the homeomorphism $\tilde{h}: \tilde{X} \to \tilde{X}$ successively on $\tilde{X}_{\text{reg}}$, $\tilde{X}_{tm}$, and $\tilde{X}_s$. Without loss of generality,

we may assume that X is weakly normal (since the weak normalization X^w of X is homeomorphic to X [1]). As observed above, X_s is a discrete subset of X and since $\eta: \tilde{X} \to X$ is a finite map, $\tilde{X}_s$ is a discrete subset of $\tilde{X}$. Also note that $\tilde{X}_s = \eta^{-1}(h(X_s))$ since, according to the lemma, $h(X_s) = X_s$.

Define $\tilde{h}': \tilde{X}_{\text{reg}} \to \tilde{X}_{\text{reg}}$ by $\tilde{h}' = \bar{\eta}^{-1} \circ h \circ \bar{\eta}$ where $\bar{\eta} = \eta|_{\tilde{X}_{\text{reg}}}: \tilde{X}_{\text{reg}} \to \tilde{X}_{\text{reg}}$ is biholomorphic. It remains to extend $\tilde{h}'$ homeomorphically across $\tilde{X}_{tm} \cup \tilde{X}_s$. Suppose that $a \in X_{tm}$. According to the lemma $(X, a) = (X_1, a) \cup \cdots \cup (X_{r(a)}, a)$ where each (X_i, a) is analytically isomorphic to $(\mathbf{C}^2, 0)$ and $(X_i, a) \cap (X_j, a)$ is homeomorphic to $(\mathbf{C}, 0)$ whenever $i \neq j$, with a similar expression for $h(X, a) = (X', h(a)) = (X'_1, h(a)) \cup \cdots \cup (X'_{r(a)}, h(a))$. Let $h_i = h|_{(X_i, a)}$. Then h_i is a homeomorphism from (X_i, a) to $(X'_i, h(a))$. Since $\eta^{-1}(a) = \{b_1, \ldots, b_{r(a)}\}$ with $\bar{\eta}_i = \eta|_{(\tilde{X}, b_i)}: (\tilde{X}, b_i) \to (X_i, a)$ biholomorphic and $\eta^{-1}(h(a)) = \{c_1, \ldots, c_{r(a)}\}$ with $\eta_i = \eta|_{(\tilde{X}, c_i)}: (\tilde{X}, c_i) \to (X'_i, h(a))$ biholomorphic, we can define $\tilde{h}: (\tilde{X}, b_i) \to (\tilde{X}, c_i)$ by the formula $\tilde{h} = \eta_i^{-1} \circ h \circ \bar{\eta}_i$. Therefore, we have extended $\tilde{h}'$ to a homeomorphism $\tilde{h}: \tilde{X} \backslash \tilde{X}_s \to \tilde{X} \backslash \tilde{X}_s$ where $\tilde{X}_s$ is a discrete subset of $\tilde{X}$. Thus we need to extend $\tilde{h}$ across $\tilde{X}_s$. If $b \in \tilde{X}_s$, then $\eta(\tilde{h}(\tilde{X}_{\text{reg}}, b))$ is a connected component of $(X \backslash S(X), h(\eta(b)))$ so the closure is an irreducible component of the germ $(X, h(\eta(b)))$. Thus the points of $\eta^{-1}(h(\eta(b)))$ are in one-to-one correspondence with the irreducible components of $(X, h(\eta(b)))$. Let $\tilde{h}(b)$ be the point corresponding to $\overline{\eta(\tilde{h}(\tilde{X}_{\text{reg}}, b))}$. This completes the construction of $\tilde{h}$ and from the method of construction it is clear that $\tilde{h}: \tilde{X} \to \tilde{X}$ is a homeomorphism, and the theorem is proved.

References

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