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A REMARK ON STRONG MAXIMUM PRINCIPLE FOR PARABOLIC AND ELLIPTIC SYSTEMS

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ABSTRACT. We give a strong maximum principle for some nonlinear parabolic and elliptic systems with convex invariant regions. We also obtain a version of the Hopf boundary lemma for the systems.

I. INTRODUCTION

The parabolic systems considered in this paper are of the form

(*)
$$\frac{\partial u}{\partial t} - D(x, t, u) \sum_{i,j=1}^{n} a_{ij}(x, t, u) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} M_i(x, t, u) \frac{\partial u}{\partial x_i} = f(x, t, u)$$

on $\Omega \times (0, T)$, where

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix},$$

Ω is a domain in $\mathbf{R}^n$, $D(x, t, u)$, and $M_i(x, t, u)$ $(i = 1, 2, \ldots, n)$ are $m \times m$ matrix-valued functions on $\Omega \times (0, T) \times R^m$, $a_{ij}(x, t, u)$ $(i, j = 1, \ldots, n)$ are real-valued functions.

Under the hypothesis that the differential operator on the left-hand side of $(*)$ is locally uniformly parabolic on $\Omega \times (0, T)$, that $(*)$ has a C^2 convex invariant region $S \subset \mathbf{R}^m$, and under some regularity conditions, we show that, for $(*)$, Weinberger's version of strong maximum principle holds, which says that *if there exists a* $(x^*, t^*) \in \Omega \times (0, T)$ *such that* $u(x^*, t^*) \in \partial S$, *then* $u(\Omega \times (0, t^*]) \subset \partial S$. Moreover, if in addition that Ω satisfies the interior sphere condition, we prove that a version of the Hopf boundary lemma holds for $(*)$.

The weak and strong maximum principle for the case that in $(*)$, $D(x, t, u) \equiv I$ and M_i $(i = 1, \ldots, n)$ are real-valued functions have been studied by Weinberger [1], the boundary point lemma, however, was not mentioned in [1]

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(see the main theorem in §3). Our basic method is the same as Weinberger's. The local defining functions of ∂S plays an important role in [1] for strong maximum principle. Instead of choosing a general defining function as in [1], we prefer the distance function of ∂S, making the proofs more geometric.

An extension of the boundary lemma was found by W. Troy [4] for non-negative solution of the elliptic system

$$\sum_{i,k=1}^{n} a_{jk}^{i}(x)\frac{\partial^2 u_i}{\partial x_j \partial x_k} + \sum_{j=1} b_j^i(x)\frac{\partial u_i}{\partial x_j} + \sum_{j=1}^{n} C_{ij}(x)u_j = 0$$

on Ω, where $i = 1, \dots, m$. $C_{ij}(x) \geq 0$ on Ω for $i \neq j$, $1 \leq i, j \leq m$.

The weak maximum principle for $(*)$ has also been studied by K. N. Chueh, C. C. Conley, and J. Smoller [2]. Their results show that for a C^1 domain $S \subset R^m$ to be an invariant region we need at least the following.

Condition (c). *S is convex and for any $u \in \partial S$, the inward unit normal $\nu(u)$ at u is a left-eigenvector of $D(x, t, u)$ and $M_i(x, t, u)$ $(i = 1, \dots, n)$, and $\nu(u) \cdot f(x, t, u) \geq 0$ for all $(x, t) \in \Omega \times (0, T)$.*

Therefore in this paper, we shall always assume that Condition (c) holds.

2. Preliminaries

All materials discussed in this section can be found in the Appendix of Chapter 14 of [3], and they are included here for the reader's convenience.

First, let's recall some classical definitions. Suppose that S is a C^2 domain in $\mathbf{R}^m$ with $\partial S \neq \phi$. For any $u \in \partial S$, let $\nu(u)$ denote the unit inner normal to ∂S at u. For a fixed $u_0 \in \partial S$, construct a coordinate system $(u_1, \dots, u_m)$ such that the u_m-axis lies in the direction $\nu(u_0)$ and the origin is at u_0. Near u_0, ∂S can be expressed by $u_m = \varphi(u_1, \dots, u_{m-1})$. Then the Gaussian curvature of ∂S at u_0 is $\det[D^2\varphi(0)]$ and the principal curvatures of ∂S at u_0 are the eigenvalues $k_1, \dots, k_{m-1}$ of the matrix $[D^2\varphi(0)]$. Now if we rotate the coordinate frame with respect to the u_m axis, we can let $u_1, \dots, u_m$ axes lie on eigenvector directions corresponding to $k_1, \dots, k_{m-1}$, respectively. We call such a new coordinate system *a principal coordinate system at* u_0. In this system $[D^2\varphi(0)] = \operatorname{diag}[k_1, \dots, k_{m-1}]$.

For $u \in \mathbf{R}^m$, the distance function d is defined by $d(u) = \operatorname{dist}(u, \partial S)$.

Lemma. *Let S be a C^k domain in $\mathbf{R}^m$, $k \geq 2$ and $\partial S \neq \varnothing$. Then there exists an open (w.r.t the topology of $\overline{S}$) subset G of $\overline{S}$ such that $G \supset \partial\Omega$, $d \in C^2(G)$, and for any $u \in G$, $\exists$ unique $y(u) \in \partial S$ such that $|u - y(u)| = d(u)$ (i.e. $u = y(u) + \nu(y(u))d(u)$), $Dd(u) = \nu(y(u))$, $1 - k_i(y(u))d(u) > 0$ $(i = 1, \dots, m-1)$ where $k_i(y(u))$ $(i = 1, \dots, m-1)$ are principal curvatures of ∂S at $y(u)$. Moreover, for $u \in G$, at a principal coordinate system at $y(u)$,*

$$[D^2 d(u)] = \operatorname{diag}\left[\frac{-k_1}{1 - k_1 d}, \dots, \frac{-k_{m-1}}{1 - k_{m-1} d}, 0\right].$$

3. The main result and its proof

In the rest of this paper, we assume that u *is a solution of* $(*)$, *and regard* D, a_{ij}, *and* M_i *in* $(*)$ *as functions of* (x, t) *only* due to the compositions.

Theorem. *Suppose that* D, a_{ij}, *and* M_i $(1 \leq i, j \leq n)$ *are locally bounded on* $\Omega \times (0, T)$, $D_{m\times m}$ *and* $(a_{ij})_{n\times n}$ *locally uniformly positive-definite on* $\Omega \times (0, T)$, *and* $f(x, t, u)$ *is Lipschitz continuous in* u *locally uniformly with respect to* (x, t) *on* $\Omega \times (0, T)$. *Assume also that there exists a* C^2 *domain* S *in* $\mathbf{R}^m$ *s.t. Condition* (c) (*in* §1) *is satisfied. Then if* $u(\Omega \times (0, T)) \subset \overline{S}$ *and there exists* $(x^*, t^*) \in \Omega \times (0, T)$ *s.t.* $u^* = u(x^*, t^*) \in \partial S$, *then* $u(\Omega \times (0, t^*]) \subset \partial S$. *Furthermore, if there exists a* $x_0 \in \partial\Omega$ *and* $0 < t_0 < T$ *s.t.* Ω *satisfies the interior sphere condition at* x_0 *and* u *is continuous at* (x_0, t_0) *with* $u(x_0, t_0) \in \partial S$, *then either* $u(\Omega \times (0, t_0]) \subset \partial S$ *or* $\nu(u(x_0, t_0)) \cdot \partial u/\partial\eta < 0$. (*if the directional derivative exists*), *where* η *is any outward pointing direction to* $\partial\Omega \times (0, T)$ *at* (x_0, t_0).

Proof. Take a bounded open neighborhood $\Omega_1 \subset \Omega$ of x^* and $0 < t_1 < t^*$ s.t. $u(\Omega_1 \times [t_1, t^*]) \subset G$ where G is defined in the Lemma of §2.

Let $\mu(x, t, \nu)$ be the eigenvalue corresponding to eigenvector ν of $D(x, t)$ and $\lambda_i(x, t, \nu)$ be the eigenvalue of $M_i(x, t)$. Then on $\Omega_1 \times [t_1, t^*]$

$$L = \frac{\partial}{\partial t} - \mu(x, t, \nu(y(u(x, t)))) \sum_{i,j=1}^{n} a_{ij}(x, t) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1} \lambda_i(x, t, \nu(y(u(x, t)))) \frac{\partial}{\partial x_i}$$

is uniformly parabolic (for definitions of ν and $y(u)$, see §2).

Let $\bar{d}(x, t) = d(u(x, t))$. Then on $\Omega_1 \times [t_1, t^*]$ we have

$$\begin{aligned} L\bar{d} &= D_u d(u) \frac{\partial u}{\partial t} - \mu(x, t, \nu(y(u))) \\ &\quad \times \sum_{i,j=1}^{n} a_{ij}(x, t) \left(\sum_{\alpha,\beta=1}^{m} \frac{\partial^2 d(u)}{\partial u_\alpha \partial u_\beta} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial u_\beta}{\partial x_j} + \sum_{\alpha=1}^{m} \frac{\partial d(u)}{\partial u_\alpha} \cdot \frac{\partial^2 u_\alpha}{\partial x_i \partial x_j} \right) \\ &\quad + \sum_{i=1}^{n} \lambda_i(x, t, \nu(y(u))) \sum_{\alpha=1}^{m} \frac{\partial d(u)}{\partial u_\alpha} \cdot \frac{\partial u_\alpha}{\partial x_i} \\ &= D_u d(u) \frac{\partial u}{\partial t} - I(x, t) - \mu(x, t, \nu(y(u))) D_u d(u) \sum_{i,j=1}^{n} a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} \\ &\quad + \sum_{i=1}^{n} \lambda_i(x, t, \nu(y(u))) D_u d(u) \frac{\partial u}{\partial x_i} \end{aligned}$$

(continues)

$$= D_u d(u)\frac{\partial u}{\partial t} - D_u d(u)D(x, t)\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + D_u d(u)\sum_{i=1}^{n} M_i \frac{\partial u}{\partial x_i}$$
$$- I(x, t)$$
$$= D_u d(u) f(x, t, u) - I(x, t),$$

where I is defined by the second equality and in the third step we use the fact that $D_u d(u) = \nu(y(u))$ and Condition (c).

Now by Condition (c) again, $\nu(y(u))f(x, t, y(u)) \geq 0$, i.e. $D_u d(y(u(x, t)))\cdot f(x, t, y(u(x, t))) \geq 0$ on $\Omega_1 \times [t_1, t^*]$. Hence we have

$$L\bar{d} \geq D_u d(u(x, t))f(x, t, u(x, t)) - D_u d(y(u(x, t)))$$
$$\cdot f(x, t, y(u(x, t))) - I(x, t)$$
$$= \tilde{c}(x, t)\cdot(u(x, t) - y(u(x, t))) - I(x, t),$$

where the R^m-vector function $\tilde{c}(x, t)$ is obtained by noticing $d \in C^2(G)$ and f is Lipschitz in u. $\tilde{c}(x, t)$ is bounded on $\Omega_1 \times [t_1, t^*]$. Since $u = y(u) + \nu(y(u))d(u)$, we have

$$L\bar{d} \geq \tilde{c}(x, t)\nu(y(u(x, t)))d(u(x, t)) - I(x, t),$$

i.e.

(1) $$L\bar{d} \geq c(x, t)\bar{d} - I(x, t) \quad \text{on } \Omega_1 \times [t_1, t^*],$$

where c is bounded.

Next, we prove $I \leq 0$ on $\Omega_1 \times [t_1, t^*]$.

Fix $(x_0, t_0) \in \Omega_1 \times [t_1, t^*]$. Since

$$\sum_{\alpha,\beta=1}^{m} \frac{\partial^2 d(u)}{\partial u_\alpha \partial u_\beta}\frac{\partial u_\alpha}{\partial x_i}\frac{\partial u_\beta}{\partial x_j}$$

is invariant under any parallel translation and rotation of u coordinate system, we assume that we work in a principle coordinate system at $y(u(x_0, t_0)) \in \partial S$. Then by the lemma

$$D_u^2 d(u(x_0, t_0)) = \operatorname{diag}\left[\frac{-k_1}{1 - k_1 d(u(x_0, t_0))}, \dots, \frac{-k_1}{1 - k_{m-1} d(u(x_0, t_0))}, 0\right]$$

where $k_1, \dots, k_{m-1}$ are the principal curvatures of ∂S at $y(u(x_0, t_0))$. Thus

$$\frac{I}{\mu}(x_0, t_0) = \sum_{i,j=1}^{n} a_{ij}(x_0, t_0)\sum_{\alpha=1}^{m-1} \frac{-k_\alpha}{1 - k_\alpha d(u(x_0, t_0))}\frac{\partial u_\alpha}{\partial x_i}(x_0, t_0)\frac{\partial u_\alpha}{\partial x_j}(x_0, t_0),$$

i.e.

(2)
$$\frac{I}{\mu}(x_0, t_0) = \sum_{\alpha=1}^{m-1} \frac{-k_\alpha}{1 - k_\alpha d(u(x_0, t_0))}\sum_{i,j=1}^{n} a_{ij}(x_0, t_0)\frac{\partial u_\alpha}{\partial x_i}(x_0, t_0)\frac{\partial u_\alpha}{\partial x_j}(x_0, t_0).$$

Since S is convex, $k_\alpha \geq 0$, $1 \leq \alpha \leq m-1$. Recall in the lemma that $1 - k_\alpha(u(u))d(u) > 0$ for $u \in G$ $(\alpha = 1, \ldots, m-1)$, so

$$\frac{I}{\mu}(x_0, t_0) \leq 0 \quad \text{on } \Omega_1 \times [t_1, t^*].$$

In view of (1), we have

$$L\bar{d} \geq c(x, t)\bar{d} \quad \text{on } \Omega_1 \times [t_1, t^*].$$

By the classical strong maximum principle, $\bar{d} \equiv 0$ on $\Omega_1 \times [t_1, t^*]$, that is $u(\Omega_1 \times [t_1, t^*]) \subset \partial S$. Thus we have proved that $u^{-1}(\partial S)$ is relatively open in $\Omega \times (0, t^*]$. Obviously $u^{-1}(\partial S)$ is relatively closed in $\Omega \times (0, t^*]$, hence $u(\Omega \times (0, t^*]) \subset \partial S$.

To prove the remaining part of the theorem, choose a bounded neighborhood Ω_2 of x_0 which is relatively open in $\overline{\Omega}$ as well as a small $\delta > 0$ such that $u(\Omega_2 \times (t_0 - \delta, t_0 + \delta)) \subset G$. In the same way as above, we have for some bounded C_0

$$L\bar{d} \geq C_0(x, t)\bar{d} \quad \text{on } \Omega_2 \times (t_0 - \delta, t_0 + \delta).$$

Thus the classical boundary point lemma gives the desired result.

Remark 1. If the strict inequality in Condition (c) holds for all $(x, t) \in \Omega \times (0, T)$, then there is no $(x^*, t^*) \in \Omega \times (0, T)$ s.t. $u(x^*, t^*) \in \partial S$.

The observations in [1] are still true for $(*)$, with slight modifications. Some of them are included in the following two remarks.

Remark 2. In the above theorem, S can be the intersection of several C^2 domains S_j which satisfy Condition (c). (In the case that S_j's meet at angles $< \pi/2$, by this paper's proof, we just need S to satisfy Condition (c).)

Remark 3. Combining (1) with $\bar{d} \equiv 0$, we have $I \geq 0$. So $I \equiv 0$. In view of (2) we have that if $k_\alpha > 0$ for all $\alpha = 1, \ldots, m-1$, $D_x u \equiv 0$. Thus we can add to the theorem that if ∂S has positive Gaussian curvature everywhere, then u is independent of x when $0 < t \leq t^*$.

Finally, concerning the elliptic systems corresponding to $(*)$, we have

Remark 4. The theorem holds for elliptic systems corresponding to $(*)$ with obvious modifications. Furthermore, it's also possible to extend the boundary point lemma for domains with corners (see [5, 6]).

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