THE OPERATOR INEQUALITY $P \leq A^*PA$

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Abstract. A short proof of the result that if $P$ is a positive compact operator and $A$ is a contraction such that $P \leq A^*PA$, then $P = A^*PA$, $\text{ran} P$ reduces $A$, and $A^*\text{ran} P$ is unitary is given.

We consider operators, i.e. elements of the algebra $B(H)$ of bounded linear transformations on a complex Hilbert space $H$ into itself. In an often referred to paper (by R. G. Douglas) it has been shown that if $P$ is a positive compact operator and $A$ is a contraction such that $P \leq A^*PA$, then $P = A^*PA$, $\text{ran} P$ reduces $A$ and $A^*\text{ran} P$ is unitary (see [1, Theorem 8 and Corollary 6.5]). The purpose of this short note is to give a proof of this result requiring little more than the well-known fact that a compact hyponormal operator is normal [3, Problem 206].

Theorem 1. Let $A$ be a contraction and $P$ be a positive operator such that $P \leq A^*PA$. If $P^{1/2}A$ is compact, then $P = A^*PA$, $\text{ran} P$ reduces $A$ and $A^*\text{ran} P$ is unitary.

Proof. Set $P^{1/2}A = M$. Then $MM^* = P^{1/2}AA^*P^{1/2} \leq P$. Hence

$$0 \leq A^*PA - P \leq A^*PA - P^{1/2}AA^*P^{1/2} = MM^* - MM^* ,$$

i.e. the operator $M$ is hyponormal. Since $M$ is compact, $M$ is normal. The normality of $M$ implies that $P = A^*PA = P^{1/2}AA^*P^{1/2}$, and hence that $A^*$ is an isometry on $\text{ran} P$ and $P$ commutes with $A$ (and so also with $A^*$). Consequently $A^*AP = A^*PA = P = PAA^*$. Hence $\text{ran} P$ reduces $A$ and $A^*\text{ran} P$ is unitary.

A more general version of the result of Douglas holds. Let $R(A^*,A): B(H) \to B(H)$ be defined by $R(A^*,A)X = A^*XA - X$, and let $R^n(A^*, A)$ denote an $n$-times application of $R(A^*, A)$.

Theorem 2. Let $A$ be contraction and $X$ be a compact operator such that $R^{n-1}(A^*, A)X \geq 0$ and $R^n(A^*, A)X \geq 0$ for some natural number $n \geq 2$. Then $A^*XA = X$, $\text{ran} X$ reduces $A$ and $A^*\text{ran} X$ is unitary.
Proof. Set $R^{n-1}(A^*, A)X = P$. Then the proof of Theorem 1 implies that $A^*PA - P = 0$ (i.e. $R^n(A^*, A)X = 0$), $\text{ran}P$ reduces $A$ and $A|\text{ran}P$ is unitary. Proceeding now as in the latter part of the proof of [2, Theorem 2(a)] (see also [4, Proof of Theorem 1']) the assertion of the statement follows.

References


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