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CENTRALIZERS OF IMMERSIONS OF THE CIRCLE

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ABSTRACT. We prove here that for every element f of an open and dense subset of immersions of the circle S^1, either the centralizer $Z(f)$ of f is trivial (i.e. f only commutes with *its* own powers) or f is topologically conjugate to a map $f_n\colon S^1 \to S^1$ given by $f_n(z) = z^n$ and, in this case, if h is the conjugacy between f and f_n then $Z(f)$ is a subgroup of $\{h^{-1} \circ \omega f_m \circ h\,;\ m \in \mathbf{N}$ and $\omega^{n-1} = 1\}$.

1. INTRODUCTION

Let $\mathrm{Imm}(S^1)$ be the group of C^∞ immersions of the circle S^1 (i.e. maps of S^1 onto itself without critical points). For $f \in \mathrm{Imm}(S^1)$, $Z(f)$ denotes the *centralizer group* of f, i.e. the set of elements in $\mathrm{Imm}(S^1)$ that commute with f. We say that f has *trivial centralizer* if $Z(f)$ is reduced to the iterates $\{f^n, n \in N\}$ of f.

We denote by f_n the immersion of S^1 given by $f_n(z) = z^n$.

The purpose of this paper is to prove the following result.

Theorem. *Let* $\mathrm{Imm}(S^1)$ *be given the* C^r *topology,* $r \in N$. *Then there is an open and dense subset* $\mathscr{U}$ *of* $\mathrm{Imm}(S^1)$ *such that for* $f \in \mathscr{U}$, *either* $Z(f)$ *is trivial or* f *is topologically conjugate to a map* f_n *and in this case, if* $h\colon S^1 \to S^1$ *is the conjugacy between* f *and* f_n *then* $Z(f)$ *is a subgroup of* $\{h^{-1} \circ \omega f_m \circ h\,;\ m \in N$ *and* $\omega^{n-1} = 1\}$.

This result is an extension to immersions of a theorem of Kopell [2, Theorem 3], who showed the triviality of the centralizer for an open and dense subset of diffeomorphisms of the circle.

The proof of the theorem is based on a result of Mañe [3] that states that structural stability is C^r generic in $\mathrm{Imm}(S^1)$.

From the theorem it follows that,

$$Z(f_n) = \{\omega f_m\,;\ m \in N\,,\ \omega^{n-1} = 1\}.$$

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Now observe that if an immersion $f: S^1 \to S^1$ is topologically conjugate to a map f_n by a conjugacy h, then f commutes with $h^{-1} \circ \omega f_m \circ h$ for all $m \in N$ and all $\omega \in S^1$ satisfying $\omega^{n-1} = 1$. Since $h^{-1} \circ \omega f_m \circ h$ is not necessarily C^∞, we were not able to prove that

$$Z(f) = \{h^{-1} \circ \omega f_m \circ h;\, m \in N,\, \omega^{n-1} = 1\}.$$

2. Proof of the theorem

We begin by recalling some basic concepts and establishing preliminary results.

Let $f: S^1 \to S^1$ be a immersion of S^1. As usual we say that $z \in S^1$ is a *periodic point* of f if $f^n(x) = x$ for some $n \in N$. In this case we say that x is a *sink* if $|(f^n)'(x)| < 1$ and a *source* if $|(f^n)'(x)| > 1$. The *basin* of a sink is defined as the set of points y such that $\lim_{n\to\infty} |f^n(x) - f^n(y)| = 0$. It is an open set containing x. We denote by $\Sigma(f)$ the complement of the union of the basins of the sinks of f. $\Sigma(f)$ is invariant under both f and f^{-1}. We say that $\Sigma(f)$ is *hyperbolic* if there exist constants $k > 0$ and $\lambda > 1$ satisfying

$$|(f^n)'(x)| > k\lambda^n \qquad \text{for all } x \in \Sigma(f) \text{ and } n > 0.$$

We note that if $h \in Z(f)$ then the set of sinks of f (sources of f) and the basins of the sinks are invariant under h.

The following lemma is basic in the proof of the theorem.

Lemma 2.1. *Let $g: S^1 \to S^1$ be a strictly monotone continuous map of S^1. If $g \circ f_k = f_k \circ g$ for some $k > 1$ then $g = \omega f_m$ where $m =$* degree g *and ω is a $(k-1)$th root of unity.*

Proof. By hypothesis, g satisfies

$$(*) \qquad g(z^k) = g(z)^k \qquad \text{for all } z \in S^1.$$

Then, if $\omega = f(1)$, we have $\omega^{k-1} = 1$ and this implies that $\omega^{-1} g$ commutes with f_k. Hence we can assume without loss of generality that $g(1) = 1$, so we have to show that

$$g(z) = z^m \qquad \text{for all } z \in S^1.$$

First we claim that g is order preserving. Otherwise, by hypothesis, there exists a point $z \in S^1$ sufficiently close to 1 and such that

$$1 < z < z^k \quad \text{and} \quad g(z^k) < g(z) < g(z)^k < 1.$$

This contradicts $(*)$ and proves the claim.

We now consider the inverse image of 1 under f_k and g. The points mapping to 1 under f_k are exactly the kth roots of unity, which can be represented by the points

$$1 < z_1 < z_1^2 < \cdots < z_1^{k-1} \qquad \text{where } z_1^k = 1.$$

Similarly, g being of degree m and strictly monotone, maps exactly m points

$$1 = p_0 < \cdots < p_{m-1}$$

to 1. We claim that

$$g(z_1^i) = (z_1^i)^m \qquad \text{for all } i = 0, 1, \dots k-1.$$

In fact, if $i = 0$, the claim is obvious because $g(1) = 1$. Suppose now, inducting on i that $g(z_1^i) = (z_1^i)^m$. Since $[z_1^i, z_1^{i+1})$ maps under f_k bijectively to S^1, there exist exactly $m+1$ points

$$z_1^i = x_0 < x_1 < \cdots < x_m = z_1^{i+1}$$

satisfying $f_k(x_j)$, $j = 0, \dots m-1$ and $f(x_m) = 1$. This and $(*)$ imply that $g(x_j)^k = 1$ and so $g(x_j) = z_1^l$ for some $l = 0, \dots k-1$. We note that between two successive elements x_j, x_{j+1} there is no other inverse image of z_1^l under g, because if $g(x) = z_1^l$ for some l, then by $(*)$

$$g(x^k) = g(x)^k = (z_1^l)^k = 1,$$

so $x^k = p_j$ for some $0 \le j \le m-1$ and this implies that $x = x_l$ for some $0 \le l \le m$. These properties together with the facts that g is order preserving and $g(z_1^i) = (z_1^i)^m$, imply that $g(x_j) = (z_1^i)^m z_1^j$. Hence

$$g(z_1^{i+1}) = (z_1^i)^m z_1^m = (z_1^{i+1})^m.$$

Similarly, since $g \circ f_k^n = f_k^n \circ g$ and $f_k^n = f_{k^n}$ for $n \in N$, we have by the arguments above that $g(z) = z^m$ for all inverse images of 1 under f_k^n. Since $\bigcup_{n \in N} f_k^{-n}(1)$ is dense in S^1, we have that $g(z) = z^m$ for all $z \in S^1$ and the lemma is proved.

Now we prove the theorem. Let

$$\beta = \{f \in \mathrm{Imm}(S^1);\ \Sigma(f) \text{ is hyperbolic}\}.$$

It follows from [3] that β is open and dense in $\mathrm{Imm}(S^1)$. Let

$$\mathscr{U}_1 = \{f \in \beta;\ \Sigma(f) \neq S^1 \text{ and } Z(f) \text{ is trivial}\}$$
$$\mathscr{U}_2 = \{f \in \beta;\ \Sigma(f) = S^1\}.$$

By [1, Theorem A], $\mathscr{U}_2$ is open in $\mathrm{Imm}(S^1)$ and if $f \in \mathscr{U}_2$ then f is topologically conjugate to a map f_k, for some $k > 1$. Let $h\colon S^1 \to S^1$ be a conjugacy between f and f_k. Then, for $g \in Z(f)$ we have that $h \circ g \circ h^{-1}$ is strictly monotone and

$$\begin{aligned}(h \circ g \circ h^{-1}) \circ f_k &= h \circ g \circ f \circ h^{-1} \\ &= h \circ f \circ g \circ h^{-1} = (h \circ f \circ h^{-1}) \circ (h \circ g \circ h^{-1}) \\ &= f_k \circ (h \circ g \circ h^{-1}).\end{aligned}$$

Thus by Lemma 2.1 $h \circ g \circ h^{-1} = \omega f_m$ where $m = \text{degree}\, g$ and ω is a $(k-1)$th root of unity. Therefore to show the theorem, it is sufficient to show that $\mathscr{U}_1$ is open in β and dense in $\beta - \mathscr{U}_2$.

We shall prove that $\mathscr{U}_1$ is dense in $\beta - \mathscr{U}_2$ by adapting a technique due to Kopell [2]. Let $f \in \beta - \mathscr{U}_2$. If degree $f = 1$ then by [2, Theorem 3], f can be arbitrarily approximated by some $\tilde{f} \in \mathscr{U}_1$, so we assume degree $f > 1$. Since $\Sigma(f)$ is hyperbolic and $S^1 - \Sigma(f)$ is nonempty, the set of sinks of f is finite and nonempty. Let $p_0^0, p_1^0, \dots, p_m^0$ be the sinks of f and let (p_i^1, p_i^2) be the component of the basin of p_i^0 containing p_i^0, $i = 0, \dots m$. Let $F = f^n$ be the first iterate of f such that $F(p_i^k) = p_i^k$, $k = 0, 1, 2$ and $i = 0, \dots m$. By making a small perturbation in f, we may assume that $F'(p_i^k) \neq F'(p_j^l)$, unless $\exists s \in N$ satisfying $f^s(p_i^k) = p_j^l$. By [2, Lemma 5], one can choose a diffeomorphism $\widetilde{F}_0 \colon [p_0^0, p_0^2] \to [p_0^0, p_0^2]$ arbitrarily close to $F/[p_0^0, p_0^2]$ such that $\widetilde{F}_0'(p_0^i) = F'(p_0^i)$, $i = 0, 2$ and $Z(\widetilde{F}_0) = \{\widetilde{F}_0^n, n \in \mathbf{Z}\}$. Let

$$\tilde{f}/s^1 - [p_0^0, p_0^2] = f/s^1 - [p_0^0, p_0^2]$$

and

$$\tilde{f}/[p_0^0, p_0^2] = \varphi \circ \widetilde{F}_0$$

where φ is the inverse of $f^{n-1}/f[p_0^0, p_0^2]$. Let $\widetilde{F} = \tilde{f}^n$. We will show that $Z(\tilde{f})$ is trivial.

First we note that since degree $\tilde{f} > 1$ and f is monotone, p_0^2 is either a limit point of $\Sigma(\tilde{f})$ or an endpoint of some interval (p_i^1, p_i^2) with $p_i^0 \neq p_0^0$. Hence by adjoining these intervals to (p_0^1, p_0^2) if necessary, we obtain an interval $I = (p_i^1, p_j^2)$ such that $(p_0^1, p_0^2) \subseteq I$. $\tilde{f}^l(I) = I$ for some $l \in N$, and the endpoints of I are limit points of $\Sigma(\tilde{f})$. It follows from the hyperbolicity of $\Sigma(\tilde{f})$ that for every neighborhood U of $\overline{I}$, $f^j(U) = S^1$ for some $j \in N$.

Now we claim that $g \in Z(\tilde{f})$ is completely determined by $g(p_0^0)$ and $g'(p_0^0)$. In fact, let $g_1, g_2 \in Z(\tilde{f})$, $g_1(p_0^0) = g_2(p_0^0)$, and $g_1'(p_0^0) = g_2'(p_0^0)$. Since the set of sinks of $\tilde{f}$ (sources of $\tilde{f}$) and the basins of the sinks are g_i-invariant, $g_1(I) = g_2(I) \neq S^1$. Hence we can choose neighborhoods U_1 and U of $\overline{I}$ with $U_1 \subset U$ and such that $\widetilde{F}/U$ and g_i/U are injective and

$$(g_2/U)^{-1} \circ g_1 \circ \widetilde{F}(x) = \widetilde{F} \circ (g_2/U)^{-1} \circ g_1(x) \qquad \text{for all } x \in U_1.$$

Since P_0^0 is a fixed point of $(g_2/U)^{-1} \circ g_1$ and $((g_2/U)^{-1} \circ g_1)'(p_0^0) = 1$, we have by [2, Lemma 1(b)] that

$$g_1(x) = g_2(x) \qquad \text{for all } x \in U_1.$$

Therefore, using the facts that $\widetilde{F}^l(U_1) = S^1$ for some $l \in N$, and $\widetilde{F}^l \circ g_i = g_i \circ \widetilde{F}^l$, $i = 1, 2$, we conclude that $g_1 = g_2$ and the claim is proved.

Now let $g \in Z(\tilde{f})$. It is easily checked that $\widetilde{F}'(p_0^0) = \widetilde{F}'(g(p_0^0))$. This together with the fact that $g(p_0^0) = p_i^0$ for some $0 \leq i \leq m$, implies $g(p_0^0) = \tilde{f}^k(p_0^0)$ for some $1 \leq k \leq n$. Since $\widetilde{F}(p_0^0) = p_0^0$, $g(p_0^0) = \tilde{f}^k \circ \widetilde{F}^j(p_0^0)$ for all $j \in N$. Hence, to show that g is a power of $\tilde{f}$, it is sufficient to show that $g'(p_0^0) = (\tilde{f}^k)'(p_0^0)(\widetilde{F}^j)'(p_0^0)$ for some $j \in N$. But, if φ_1 is the inverse of $\tilde{f}^k/[p_0^0, p_0^2]$ then $\varphi_1 \circ g/[p_0^0, p_0^2] \in Z(\widetilde{F}_0)$, so $(\varphi_1 \circ g)'(p_0^0) = (\widetilde{F}^j)'(p_0^0)$ for some $j \in N$. Hence we are done.

The proof that $\mathscr{U}_1$ is open in β uses similar arguments and the Lemma 4 of [2].

References

1. M. V. Jacobson, *On smooth mappings of the circle into itself*, Math. USSR-Sb. **14** (1971), 163–168.

2. N. Kopell, *Commuting diffeomorphisms*, Global Analysis, Proc. Sympos. Pure Math. **14** (1970), 165–184.

3. R. Mañe, *Hyperbolicity, sinks and measure in one dimensional dynamics*, Comm. Math. Phys. **100** (1985), 495–524.

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