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LOWER BOUNDS FOR THE EXTRINSIC TOTAL CURVATURES OF A SPACE-LIKE CODIMENSION 2 SURFACE IN MINKOWSKI SPACE

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ABSTRACT. There are three invariant curvature functions defined on any smooth space-like 2-surfaces in four-dimensional Minkowski space. (If the surface lies in a Euclidean hyperplane then the functions agree with H^2, K^2, and $(H^2 - K)^2$. For each of these functions we show that there exists a space-like immersion of any oriented compact (or noncompact complete) surface with associated total curvature arbitrarily small.

A two dimensional oriented surface M in four dimensional Minkowski space which inherits a positive definite metric is called space-like. Such a surface carries three functions $\mathbf{H}$, $\mathbf{K}$, and $\mathbf{U}$ that are determined by the first and second fundamental forms. (If the surface lies in a Euclidean hyperplane then these functions reduce to H^2, K^2, and (H^2-K^2) respectively.) It is natural to ask for the infimum of the respective total curvatures $\int_M |\mathbf{H}|\,dA$, $\int_M |\mathbf{K}|^{1/2}\,dA$ and $\int_M \mathbf{U}^{1/2}\,dA$ over the set of embedded compact space-like surfaces. In this paper we show that there exists an embedded smooth compact orientable surface M_g of genus $g \geq 0$ with $\int_{Mg} |\mathbf{H}|\,dA$ arbitrarily small (Theorem 5). In the process we relate $\mathbf{H}$ to the split mean curvatures in the manner of reference [2]. We then show that there exist compact (and complete noncompact) embedded orientable surfaces of arbitrary genus with either $\mathbf{K}$ or $\mathbf{U}$ identically zero, (Theorems 6 and 7). The author would like to thank P. Ehrlich, R. Howard, T. K. Milnor and R. Penrose for inspiration.

Throughout this paper $\mathbf{M}^4$ will denote Minkowski space, the real four dimensional vector space equipped with bilinear form $\langle\ ,\ \rangle$ of type $(3, 1)$, (i.e., the normal form has 3 plus signs and 1 minus sign). We will assume that $\mathbf{M}^4$ is oriented and time oriented (i.e., a 4-volume form dV, and future time-like vector field Fut, have been chosen.) We will write $LC = \{v \in \mathbf{M}^4 | \langle v , v\rangle = 0\}$ to denote the light cone in Minkowski space. A space-like surface in $\mathbf{M}^4$ is

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a smooth immersion of a smooth compact oriented 2-manifold $k\colon M \to \mathbf{M}^4$, such that the induced metric $k^*\langle\ ,\ \rangle$ is positive definite. It follows that the normal bundle $M^\perp \to M$ is a trivial rank two bundle with fibres carrying a type $(1, 1)$ metric. Now choose an orthogonal splitting of Minkowski space $\mathbf{M}^4 = \mathbf{E}^3 + \mathbf{E}^-$ with projections $t\colon \mathbf{E}^3 + \mathbf{E}^- \to \mathbf{E}^-$ and $\pi_{\mathbf{E}}\colon \mathbf{E}^3 + \mathbf{E}^- \to \mathbf{E}^3$ and let U_s^i, $i = F, P$ denote the two unique sections of $M^\perp$ which satisfy the following:

(E1)
(a) $\langle U_s^i, U_s^i\rangle = 0$, $\quad i = F, P$,
(b) $t_*(U_s^i(x))$ is of unit length for all $x \in M$, $\quad i = F, P$,
(c) $\langle U_s^F, \mathrm{FUT}\rangle < 0$,
(d) $dV(-, -, U_s^p, U_s^F) \equiv dA(-, -)$,

(Here dA is the induced area form on M and equivalence is in the sense of orientations. The subscript s refers to the choice of splitting.)

We also have the splitting dependent function $\frac{1}{\sqrt{2}}\langle U_s^F, U_s^P\rangle\colon M \to R$, which we denote by TILT_s, and which satisfies:

(E2)
(a) $0 < \mathrm{TILT}_s \leq 1$
(b) TILT_s is constant iff $\mathrm{TILT}_s = 1$ and M lies in a fiber of $t\colon \mathbf{E}^3 + \mathbf{E}^- \to \mathbf{E}^-$.

See Lemma 2 in [1].

Now for any choice of splitting $\mathbf{M}^4 = \mathbf{E}^3 + \mathbf{E}^-$ we get two split second fundamental forms $\Pi_s^i\colon TM \times TM \to \mathbf{R}$, $i = F, P$ defined by $\Pi_s^i(X, Y) = \langle \nabla_X U_s^i, Y\rangle$ where X, Y are vector fields on M and ∇ is the ambient connection. We define the *split curvatures* by $K_s^i = \frac{\mathrm{DET}\,\Pi_s^i}{\mathrm{DET}\,\mathrm{I}}$, $i = F, P$ and *split mean curvatures* by $H_s^i = \frac{1}{2}\frac{\mathrm{TR}\,\Pi_s^i}{\mathrm{DET}\,\mathrm{I}}$, $i = F, P$ where I is the first fundamental form on M.

For any normal vector $u_x \in M_x$, we have an endomorphism of T_xM defined by $v_x \to$ (tangential projection of $\nabla_v u$ at x). There are unique global sections R^F, R^P of M that satisfy $\langle R^F, R^P\rangle = \sqrt{2}$ and a,c, and d of (E1) above. For such a pair we have the *quadric mean curvature* $\mathbf{H} = \frac{1}{4}\mathrm{TR}(\nabla R^F)\cdot\mathrm{TR}(\nabla R^P)$ and the *quartic curvature* $\mathbf{K} = \mathrm{DET}(\nabla R^F)\cdot\mathrm{DET}(\nabla R^P)$. Observe that if we fix a splitting $\mathbf{M}^4 = \mathbf{E}^3 + \mathbf{E}^-$, then

$$\text{(E3)}\qquad \mathbf{H} = \frac{H_s^F H_s^P}{\mathrm{TILT}_s}, \text{ and} \qquad \mathbf{K} = \frac{K_s^F K_s^P}{(\mathrm{TILT}_s)^2}$$

From these splitting dependent descriptions of $\mathbf{H}$ and $\mathbf{K}$ it is apparent that

$$\text{(E4)}\qquad \mathbf{U} = \frac{(K_s^F - (H_s^F)^2)(K_s^P - (H_s^P)^2)}{(\mathrm{TILT}_s)^2} \geq 0$$

is also an invariant, which we will call the *umbilic curvature.* All three functions $\mathbf{H}, \mathbf{K}, \mathbf{U}: M \to \mathbf{R}$ are congruence invariants for (M, k).

Proposition 1. *Given* (M, k), *then for all splittings* $\mathbf{M}^4 = \mathbf{E}^3 + \mathbf{E}^-$,

$$\int_M |\mathbf{H}|\, dA \le \left[\sup \frac{1}{2\,\mathrm{TILT}_s}\right] \int_M (H_s^F)^2 + (H_s^P)^2\, dA .$$

Further if equality holds for some splitting then (M, k) *is an immersion into a space-like hyperplane* $\mathbf{E}^3 \subset \mathbf{M}^4$. (*The supremum is taken over all splittings* $\mathbf{M}^4 = \mathbf{E}^3 + \mathbf{E}^-$).

Proof. Choose a splitting. Then (E3) implies that

$$2|\mathbf{H}| \le \frac{1}{\mathrm{TILT}_s}\left[(H_s^F)^2 + (H_s^P)^2\right],$$

from which follows the inequality

$$2\int_M |\mathbf{H}|\, dA \le \int_M \frac{(H_s^F)^2 + (H_s^P)^2}{\mathrm{TILT}_s}\, dA \le \left[\sup \frac{1}{\mathrm{TILT}_s}\right] \int_M (H_s^F)^2 + (H_s^P)^2\, dA .$$

Since equality implies (TILT_s) is constant (E2b) implies our claim.

Theorem 2. *Given* (M, k) *then for all splittings* $\mathbf{M}^4 = \mathbf{E}^3 + \mathbf{E}^-$,

$$8\pi \le \left[\sup \frac{1}{\mathrm{TILT}_s}\right] \int_M (H_s^F)^2 + (H_s^P)^2\, dA .$$

Further if equality holds for some splitting then $M = S^2$ *and* k *embeds* S^2 *as a round sphere in a space-like hyperplane* $\mathbf{E}^3 \subset \mathbf{M}^4$.

Proof. Recall that for any splitting we have:

$$K_s^i \le (H_s^i)^2 \qquad i = F, P$$

with equality at $x \in M$ if and only if x is an umbilic point for Π_s^i, $i = F, P$. Now with (E7) of [1] we have that

$$\text{(E5)}\quad \begin{aligned} 8\pi &\le \int_{\{K_s^F > 0\}} K_s^F\, dA + \int_{\{K_s^P > 0\}} K_s^P\, dA \\ &\le \int_{\{K_s^F > 0\}} (H_s^F)^2\, dA + \int_{\{K_s^P > 0\}} (H_s^P)^2\, dA \\ &\le \int_M (H_s^F)^2 + (H_s^P)^2\, dA \\ &\le \left[\sup \frac{1}{\mathrm{TILT}_s}\right] \int_M (H_s^F)^2 + (H_s^P)^2\, dA \end{aligned}$$

Finally if we have equality, then $\mathrm{TILT}_s = 1$, hence (M, k) is an immersion into an $\mathbf{E}^3$ factor of the splitting with $\mathbf{E}^3$-mean curvature given by $H_s^F = H_s^P$ and Gaussian curvature given by $K_s^F = K_s^P$. Thus (M, k) is a totally umbilic surface in a $\mathbf{E}^3 \subset \mathbf{M}^4$.

Next we examine how the split mean curvatures behave individually.

Theorem 3.

(a) *Given* (M, k) *with* $M = S^2$, *then for all splittings* $\mathbf{M}^4 = \mathbf{E}^3 + \mathbf{E}^-$

$$4\pi \leq \int_{S^2} (H_s^i)^2 \, dA \qquad i = F, P$$

Further equality for $i = F$ *or* P *implies that* k *embeds* S^2 *in a light cone* $LC \subset \mathbf{M}^4$. *Equality in both* $i = F$ *and* P *implies that* (M, k) *is congruent to a round* S^2 *in a space-like hypersurface* $\mathbf{E}^3 \subset \mathbf{M}^4$.

(b) *Given* $\varepsilon > 0$ *and an integer* $g > 0$, *then there exists an orientable compact space-like embedded surface of genus* g *and a splitting* $\mathbf{M}^5 = \mathbf{E}^3 + \mathbf{E}^-$ *such that for* $i = F$ *or* P

$$0 \leq \int_{M_g} (H_s^i)^2 \, dA \leq \varepsilon .$$

(Note that because of (E5) above this holds for only *one* of the split mean curvatures.)

Proof. (a) Referring to Theorem 1 of [1] we have that

$$4\pi \leq \int_{\{K_s^i > 0\}} K_s^i \, dA, \qquad i = F, P$$

where the integration is over the subset of M where $K_s^i > 0$. As a consequence we have that for $i = F, P$,

$$\begin{aligned} 4\pi &\leq \int_{\{K_s^i > 0\}} K_s^i \, dA \\ &\leq \int_{\{K_s^i > 0\}} (H_s^i)^2 \, dA \leq \int_{s^2} (H_s^i)^2 \, dA . \end{aligned}$$

Now suppose there is equality for $i = F$ or P. From (E6) of [2] it follows that *as conformal structures* on M, Π_s^i for (M, k) agrees with $\overline{\Pi}$ for the light-like hypersurface $\mathrm{LL}^i(M, k)$. Thus if we choose any (affine) space-like hyperplane $\mathbf{E}^3 \subset \mathbf{M}^4$, at any immersed point in $\{\mathbf{E}^3 \cap \mathrm{LL}^i(M, k)\}$ this intersection is locally an umbilic surface in $\mathbf{E}^3$, and hence is a piece of a round sphere or of a plane. Since we are assuming M is compact, we will have our result if we show that there exists an $\mathbf{E}^3 \subset \mathbf{M}^4$ with intersection $\{\mathbf{E}^3 \cap \mathrm{LL}^i(M, k)\}$ consisting of only immersive points (i.e., $\mathrm{LL}^i(M, k)$ is an icon). To prove this choose a hyperplane $\widetilde{\mathbf{E}}^3 \subset \mathbf{M}^4$ so that the intersection $\{\widetilde{\mathbf{E}}^3 \cap \mathrm{LL}^i(M, k)\}$ contains an immersive point. Recall that $\mathrm{LL}^i(M, k) \xrightarrow{\pi^i} M$ is fibered by null lines and each fiber intersects the chosen $\widetilde{\mathbf{E}}^3$ in a unique point. (See [1] for details.) Since $k\colon M \to \mathbf{M}^4$ has image, which is a section of this fibering of $\mathrm{LL}^i(M, k)$, we now have smooth maps

$$\begin{array}{c} S^2 = M \xrightarrow{k} \mathrm{LL}^i(M, k) \subset \mathbf{M}^4 \\ \big\downarrow p \\ \widetilde{\mathbf{E}}^3 \subset \mathbf{M}^4 \end{array}$$

where $p \circ k(x)$ is intersection point of the π^i-fibered through $k(x)$ and the chosen $\widetilde{\mathbf{E}}^3$. We know the immersive points of $p \circ k$ are open in $S^2 = M$, we need only show that they are closed. Let $x_n \to x$ be a convergent sequence of points in S^2 with $p \circ k$ immersive at x_n. Now consider the $\mathbf{E}^3$ parallel to the above chosen $\widetilde{\mathbf{E}}^3$ that contains $k(x)$. We know that $\{\mathbf{E}^3 \cap \mathrm{LL}^i(M, k)\} = V$ is a piece of a sphere or a plane. Hence $(\pi^i)^{-1}(\pi^i(V))$ is a subset of a light cone or a null 3-plane. But for any affine $\mathbf{E}^3 \subset \mathbf{M}^4$, the intersection of $\mathbf{E}^3$ with a light cone or a null 3-plane consists entirely of immersive points or entirely of nonimmersive points. Thus the intersection of $(\pi^i)^{-1}(\pi^i(V)) \subset \mathrm{LL}^i(M, k)$ with the chosen $\widetilde{\mathbf{E}}^3$ contains $p \circ k(x_n)$ (for n sufficiently large). These are immersive points for $p \circ k$ and hence immersive points for the intersection with $\widetilde{\mathbf{E}}^3$. Since $p \circ k(x)$ lies in this intersection it is also an immersive point. We conclude $\mathrm{LL}^i(M, k)$ is congruent to a light cone. For the last claim in Theorem 3 (a) we now know that both $\mathrm{LL}^i(M, k)$ are congruent to light cones. Hence (M, k) is space-like, compact, and lies in their intersection. It must be a round $S^2 \subset \mathbf{E}^3 \subset \mathbf{M}^4$.

(b) The construction of (M_g, k) is but a slight modification of the construction presented in Appendix 2 of [1]. Consider a fixed splitting $\mathbf{M}^4 = \mathbf{E}^3 + \mathbf{E}^-$ with projection $t\colon \mathbf{E}^3 + \mathbf{E}^- \to \mathbf{E}^-$ and the torus of revolution (T^2, k) obtained by rotating an "embedded Figure 8" about a $\mathbf{M}^2$ in $\mathbf{M}^4$ so tha a linear segment in the figure 8 generates a cylinder in $\mathbf{E}^3$; and the S^2 valued Gauss maps are not surjective. For this torus and splitting $H_s^i = \frac{1}{2}\frac{\mathrm{TR}\langle \nabla U_s^i, \rangle}{\mathrm{DET\,I}}$, $i = F, P$. Now because the Gauss maps are not surjective, there exist isometries $b_\theta\colon \mathbf{M}^4 \to \mathbf{M}^4$ such that for all positive integers $n > 0$, there exist θ_n so that $n < \mathrm{INF}\,|t_*(b_{\theta_n} \circ U_s^F)|$ where the infimum is over points of the torus. Since b_{θ_n} is an isometry we have that $(T^2, b_{\theta_n} \circ k)$ has $U_{s\theta_n}^F = \lambda(b_n \circ U_s^F)$ where $\lambda > \frac{1}{n}$ is a function on T^2 and $U_{s\theta_n}^F$ is the vector field satisfying (E1) above relative to the fixed splitting. We have that,

$$H_{s\theta_n}^F = \frac{\mathrm{TR}\langle \nabla U_{s\theta_n}^F, -\rangle}{\mathrm{DET\,I}} < \frac{1}{n}\frac{\mathrm{TR}\langle \nabla U_s^F, -\rangle}{\mathrm{DET\,I}} < \frac{H_s^F}{n}$$

and hence a torus with $\int_{T^2} (H_s^F)^2\, dA$ arbitrarily small. Note that the key step of constructing b_{θ_n} requires only that the Gauss map be nonsurjective. Thus to construct higher genus examples take two copies of (T^2, k) and shrink one copy so that the two copies do not intersect, and the cylinders in $\mathbf{E}^3$ mentioned at the outset are concentric. We can connect these two cylinders in the $\mathbf{E}^3$ with catanoidal necks in such a way that removing the $\mathbf{E}^3$-minimal parts of the necks results in two punctured tori with $\mathbf{M}^4$-Gauss maps nonsurjective. If we denote this connected sum by (M_g, k) then for any splitting $\mathbf{M}^4 = \mathbf{E}^3 + \mathbf{E}^-$, both the H_s^i, $i = F, P$ will vanish on the $\mathbf{E}^3$-minimal part of the necks. Thus we may repeat the above argument on $(M_g - \{H_s^F = 0\}, b_{\theta_n} \circ k)$ and we have surfaces which satisfy our claim for $i = F$. The same argument adapts to construct an example for $i = P$.

By a *space-like graph* in $\mathbf{M}^4$ we mean a smooth space-like hypersurface which, relative to some (hence any) splitting of $\mathbf{M}^4 = \mathbf{E}^3 + \mathbf{E}^-$ is globally the graph of a function $f: \mathbf{E}^3 \to \mathbf{E}^-$.

Theorem 4. *If* (M, k) *is an embedding into a space-like graph, then for all splittings of* $\mathbf{M}^4 = \mathbf{E}^3 + \mathbf{E}^-$

$$4\pi \le \int_M (H_s^i)^2 \, dA, \qquad i = F, P.$$

Further equality for $i = F$ or P implies that $M = S^2$ so that Theorem 3, Part (a) applies.

Proof. Recall from the proof of Theorem 9 in [1] that for (M, k) a surface embedded in a graph we must have

$$4\pi \le \int_{\{K_s^i > 0\}} K_s^i \, dA, \qquad i = F, P.$$

(This is a restatement of the first line in the proof of Theorem 9.) Now for the case of equality we must have by Theorem 1 of [1] that $M = S^2$.

Theorem 5. *Given* $\varepsilon > 0$ *and an integer* $g \ge 0$, *there exists an orientable compact embedded space-like surface of genus* g *with,*

$$0 < \int_{M_g} |\mathbf{H}| \, dA < \varepsilon.$$

Proof. We first construct an example with $g = 0$. Take two round spheres S_1^2 and S_2^2 in $\mathbf{E}^3 \subset \mathbf{M}^4$ and smoothly connect them with a "Hopf neck" $HN \subset \mathbf{E}^3$ of small total squared $\mathbf{E}^3$-mean curvature to yield an embedded sphere, $S^2 \# S^2$. This can be done as in Figure 1 so that the transition region between round sphere and Hopf neck consist of a pair of planar parallel anuli in $\mathbf{E}^3$. (For details concerning the Hopf neck see Lemma 6b in [3].) Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be the parallel planes in which these annuli lie. Now extend the $S^2 \# S^2 \subset \mathbf{E}^3$ to a Light-like hypersurface $\mathrm{LL}(S^2 \# S^2)$ in $\mathbf{M}^4$ so that the vertex of round spherical parts lie in the "future" of the $\mathbf{E}^3$ fixed at the outset. Similarly extend the round spheres S_1^2 and S_2^2 to light cones LC_1 and LC_2 so that their vertices are in the future of the $\mathbf{E}^3$. Next extend the 2-planes $\mathbf{P}_1$ and $\mathbf{P}_2$ to null 3-planes $\mathbf{N}_1^3$

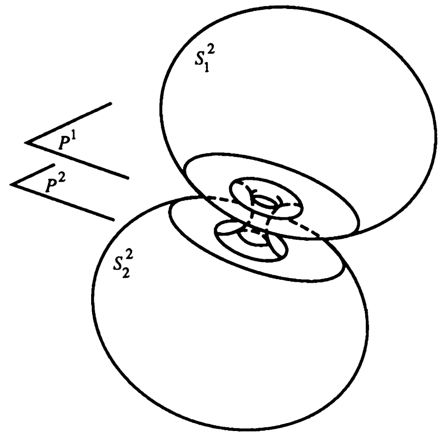

FIGURE 1

and $\mathbf{N}_2^3$ so that $\mathbf{N}_\alpha^3$ intersects the future of LC_α in a bounded set. Let $\overline{\mathbf{N}}_\alpha^{3F}$, $\alpha = 1, 2$ denote the closure of the subsets of $\mathbf{N}_\alpha^3$ that lie in the future of $\mathbf{E}^3$ and let $D_\alpha = \mathrm{LL}(S^2 \# S^2) \cap \overline{\mathbf{N}}_\alpha^{3F}$, $\alpha = 1, 2$. These D_α are embedded space-like 2-disks. Thus by construction $\{D_1 \cup HN \cup D_2\}$ is an embedded space-like two-sphere that intersects the fixed $\mathbf{E}^3$ in the Hopf neck along with its pair of planar annuli. Now since each D_α lies in a null 3-plane $\mathbf{H}$ vanishes on both D_α. Thus the support of $\mathbf{H}$ lies entirely in the Hopf neck where it agrees with the $\mathbf{E}^3$-mean curvature squared. We have our sphere. To construct higher genus examples we need only connect the parallel annuli with additional Hopf necks with small total squared $\mathbf{E}^3$-mean curvature and we are finished.

We now construct space-like surfaces with $\mathbf{K}$ and $\mathbf{U}$ identically zero. We remark that the existence of $\mathbf{K}$-flat surfaces is relevant to the conjecture discussed in Appendix 3 of [1] in that these surfaces indicate that there need not exist a point $x \in M$ where both split curvatures are positive $K_s^i(x) > 0$ $i = F, P$.

Theorem 6. *Given $g \geq 0$, there exist orientable embedded compact (or noncompact complete) space-like surfaces of genus g with $\mathbf{K} = 0$.*

Proof. Consider any two $\mathbf{E}^3$-flat surfaces $\bar{k}^i$: $\mathrm{FLT}^i \to \mathbf{E}^3 \subset \mathbf{M}^4$, $i = F, P$ that intersect transversely on an immersed circle $\widetilde{S}^1 \to \mathrm{FLT}^F \cap \mathrm{FLT}^P$. (The reason for indexing these surfaces by F and P will be apparent below.) Next choose $\mathbf{E}^3$-normals for each surface n^i: $\mathrm{FLT}^i \to S^2 \subset \mathbf{E}^3$, $i = F, P$. As in the proof of Proposition 6 of [2] we use these $\mathbf{E}^3$-normals to construct two light-like hypersurfaces j^i: $\mathrm{LL}^i(\mathrm{FLT}^i) \to \mathbf{M}^4$, $i = F, P$. Now two null 3-planes in $\mathbf{M}^4$ intersect in an $\mathbf{E}^2 \subset \mathbf{M}^4$ or coincide. Since FLT^i intersect transversely in $\mathbf{E}^3$ we have that $\mathrm{LL}^F(\mathrm{FLT}^F)$ intersects $\mathrm{LL}^P(\mathrm{FLT}^P)$ transversely in $\mathbf{M}^4$. Thus near the intersecting circle we have that $\mathrm{LL}^F(\mathrm{FLT}^F) \cap \mathrm{LL}^P(\mathrm{FLT}^P)$ is an immersed space-like cylinder $\widetilde{S}^1 \times (-\delta, \delta)$. (It will be embedded if the $\mathrm{FLT}^i \subset \mathbf{E}^3$, $i = F, P$ are embedded.) It follows that this small cylinder is transverse to the fibers of both π^i: $\mathrm{LL}^i(\mathrm{FLT}^i) \to \mathrm{FLT}^i$. We may view the original FLT^i $i = F, P$ as "zero sections" of π^i and observe that the intersection cylinder is transverse to both of these zero sections in $\mathrm{LL}^i(\mathrm{FLT}^i)$. Thus we may extend the $\widetilde{S}^1 \times (-\delta, \delta)$ so that one boundary edge $\widetilde{S}^1 \times \{-\delta\}$ extends so as to agree with the zero section of π^F outside of a compact set and the other boundary edge $\widetilde{S}^1 \times \{\delta\}$ extends so as to agree with the zero section of π^P outside a compact set. We may construct this extension so that it is everywhere transverse to (at least) one of the fibrations π^i, $i = F, P$, thence this *extended intersection* $(\mathrm{FLT}^F \# \mathrm{FLT}^P, k)$ is a space-like surface in $\mathbf{M}^4$.

In Figure 3 we illustrate our construction and include the image of the extended intersection under the projection $\pi_{\mathbf{E}}: \mathbf{M}^4 \to \mathbf{E}^3$. Now (E10) of [2] tells us that any section of either $\mathrm{LL}^i(\mathrm{FLT}^i)$ $i = F, P$ must have $\mathbf{K}$ identically zero. Using the construction on flat surfaces as represented by Figures 2 and 3 we can build spheres or tori and take connected sums so that the resulting surfaces have $\mathbf{K}$ identically zero. We have our surfaces.

Theorem 7. *Given $g \geq 0$ there exists an orientable embedded compact (or non-compact complete) space-like surface of genus g with $\mathbf{U} = 0$.*

Proof. The proof parallels that of Theorem 6 with the modification that flat surfaces in $\mathbf{E}^3$ are replaced by round spheres S_i^2, $i = F, P$ in $\mathbf{E}^3$. The key point is again (E10) of [2], which in this context tells us that any section of either $\mathrm{LL}^i(S_i^2)$ $i = F, P$ must have $\mathbf{U}$ identically zero.

It should be apparent to the reader that these *subumbilic* immersions are highly nonrigid. So that $\mathbf{F}(M, k) = \int_M U^{1/2}\, dA$ defines a geometric functional

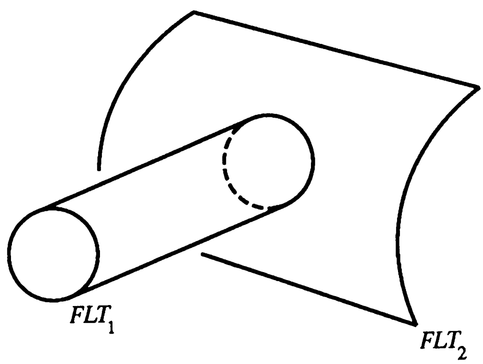

FIGURE 2

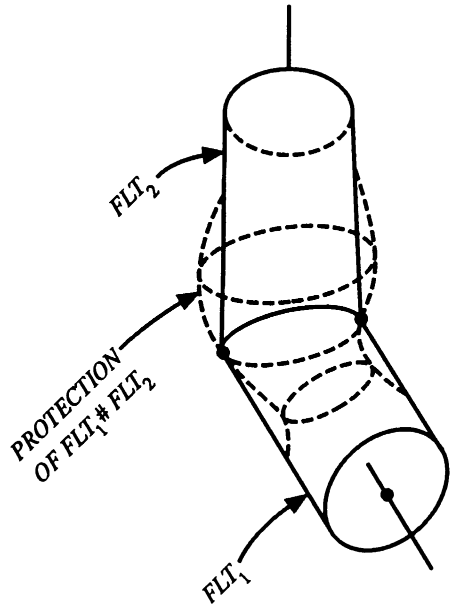

FIGURE 3

on $\mathrm{Immer}^{\infty}(M, \mathbf{M}^4)$ with highly degenerate second variation at the minima constructed above.

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