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TOPOLOGICAL RATIONALIZATION OF A CLASS OF MEROMORPHIC FUNCTIONS

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ABSTRACT. We prove that a class of germs at $0 \in \mathbf{C}^2$ of meromorphic functions can be transformed, via homeomorphisms, into rational functions.

0. INTRODUCTION

We prove in this note that if $M(x, y)$ is the germ at $0 \in \mathbf{C}^2$ of a pure meromorphic function whose blow-up $\widetilde{M} = M \circ E$, where E is the blow-up of $\mathbf{C}^2$ at 0, has no essential singularities on $E^{-1}(0)$, then M is topologically conjugate to a rational function $R(x, y)$. That is, there is a homeomorphism Φ of a neighborhood of $0 \in \mathbf{C}^2$ such that $M \circ \Phi = R$.

D. Cerveau and J. F. Mattei have studied [1] finite determination for multiform functions. In particular, they prove rationalization for germs of meromorphic functions with critical set reduced to $\{0\}$. They also show that germs at $0 \in \mathbf{C}^2$ of holomorphic functions are conjugate to polynomials.

We begin this note by recalling briefly the blow-up of $\mathbf{C}^2$ at 0, and then we define the class of meromorphic functions that we will be dealing with. Next, we prove an intermediate result: a particular conjugation of germs of analytic varieties. Finally, we state and prove the main result.

1. DEFINITIONS

In this section, we recall very briefly the blow-up of $\mathbf{C}^2$ at 0 (see [4]) and define a class of meromorphic functions, called here class A. Some examples will follow.

The blow-up of $\mathbf{C}^2$ at 0 is the subset of $\mathbf{C}^2 \times \mathbf{CP}^1$, where $\mathbf{CP}^1$ is the complex projective space of dimension 1, given as

$$\mathbf{C}_0^2 = \left\{(p, [p]) \in \left(\mathbf{C}^2 - \{0\}\right) \times \mathbf{CP}^1\right\} \cup \left(\{0\} \times \mathbf{CP}^1\right),$$

where $[\ ]: \mathbf{C}^2 - \{0\} \mapsto \mathbf{CP}^1$ is the quotient map. It is shown, in [4] for example, that $\mathbf{C}_0^2$ is a two-dimensional complex manifold. Moreover, $\mathbf{C}_0^2$ is covered by

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two charts (x, t), (s, y) (t, s are the canonical coordinates of $\mathbf{CP}^1$) related by

$$s = \frac{1}{t} \quad \text{and} \quad y = xt.$$

The first projection $E: \mathbf{C}_0^2 \mapsto \mathbf{C}^2$, called the blow-up map, is given in the charts (x, t), (s, y) by

$$E(x, t) = (x, xt) \quad \text{and} \quad E(s, y) = (sy, y).$$

E is a biholomorphism from $\mathbf{C}_0^2 - E^{-1}(0)$ onto $\mathbf{C}^2 - \{0\}$. The exceptional divisor $E^{-1}(0)$ is identified with $\mathbf{CP}^1$.

A meromorphic function M on a two-dimensional complex manifold $\mathscr{M}$ is said to have an *essential singularity* at $p_0 \in \mathscr{M}$ if, near p_0, the function M is expressed as a quotient α/β, with α, β germs at p_0 of coprime holomorphic functions and $\alpha(p_0) = \beta(p_0) = 0$. Equivalently, M has an essential singularity at p_0 if, in a neighborhood of p_0, all the level sets of M pass through p_0. In this case, as in [1], we say that M is a *pure meromorphic* function at p_0.

Definition 1.1. A germ at $0 \in \mathbf{C}^2$ of a pure meromorphic function $M(x, y)$ is said to be in the *class* A if the map $\widetilde{M} = M \circ E: \mathbf{C}_0^2 \mapsto \overline{\mathbf{C}}$, defined in a neighborhood of $E^{-1}(0) = \mathbf{CP}^1 \subset \mathbf{C}_0^2$, has no essential singularities.

Remark 1.1. It is easy to see, as a consequence of the desingularization of foliations defined by level sets of meromorphic functions (see [3] for example), that if M is the germ at $0 \in \mathbf{C}^2$ of a pure meromorphic function, then there is a proper holomorphic map $\Pi: \mathscr{M} \mapsto \mathbf{C}^2$, obtained by composition of blow-ups, such that the lift $M \circ \Pi$ of M to $\mathscr{M}$ has no essential singularities. Hence, the class A is the set of all meromorphic functions for which $\mathscr{M} = \mathbf{C}_0^2$.

Example 1. The function

$$M(x, y) = \frac{x^3 + xy^2 + \alpha(x, y)}{y^3 + \beta(x, y)},$$

where α, β are holomorphic functions of order ≥ 4, is in the class A. The blow-up of M is given in the charts (x, t), (s, y) respectively by

$$\widetilde{M}(x, t) = \frac{1 + t^2 + \left(\alpha(x, tx)/x^3\right)}{y^3 + \left(\beta(x, tx)/x^3\right)}, \qquad \widetilde{M}(s, y) = \frac{s^3 + s + \left(\alpha(sy, y)/y^3\right)}{1 + \left(\beta(sy, y)/y^3\right)}.$$

So $\widetilde{M}$ has no essential singularities on $\mathbf{CP}^1 = \{x = 0\} \cup \{y = 0\}$.

Example 2. The meromorphic function $M(x, y) = (y^2 + x^3)/xy$ is not in the class A. Its blow-up $\widetilde{M}(x, t) = (t^2 + x)/t$ has an essential singularity at $(x = 0, t = 0)$. A blow-up of $\mathbf{C}_0^2$ at $(x = 0, t = 0)$ will produce a meromorphic function with no essential singularities.

Remark 1.2. The functions of the form x^r/y^s, where r, s are positive integers and $r > s$, are not in the class A. However, we characterized in [3] all meromorphic functions that are conjugate to the rational functions of the form $R(x^r/y^s)$, where R is a rational function on $\overline{\mathbf{C}}$.

The following proposition characterizes the class A.

Proposition 1.1. *Let $M(x, y) = \alpha(x, y)/\beta(x, y)$ be the germ at $0 \in \mathbf{C}^2$ of a pure meromorphic function and let*

$$\alpha(x, y) = \sum_{i \geq n} P_i(x, y) \quad \text{and} \quad \beta(x, y) = \sum_{j \geq m} Q_j(x, y)$$

be the Taylor expansions of α and β (P_i and Q_j are homogeneous polynomials with respective degrees i and j). Then M is in the class A if and only if $n = m$ and P_n, Q_n are coprime.

Proof. We first show that if $n = m$ and P_n, Q_n are coprime, then M is in the class A. The blow-up $\widetilde{M} = M \circ E$ of M is given in the charts (x, t), (s, y) by

$$\widetilde{M}(x, t) = \frac{P_n(1, t) + xP_{n+1}(1, t) + \dots}{Q_n(1, t) + xQ_{n+1}(1, t) + \dots},$$
$$\widetilde{M}(s, y) = \frac{P_n(s, 1) + yP_{n+1}(s, 1) + \dots}{Q_n(s, 1) + yQ_{n+1}(s, 1) + \dots},$$

Since $P_n(1, t)$ and $Q_n(1, t)$ (resp. $P_n(s, 1)$ and $Q_n(s, 1)$) have no common root, then $\widetilde{M}$ has no essential singularities on $E^{-1}(0)$.

Conversely, let us show that if M is in the class A, then $n = m$ and P_n, Q_n are coprime. By contradiction, first if $n = m + k > m$, then

$$\widetilde{M}(x, t) = \frac{x^k\left(P_{m+k}(1, t) + xP_{m+k+1}(1, t) + \dots\right)}{Q_m(1, t) + xQ_{m+1}(1, t) + \dots},$$
$$\widetilde{M}(s, y) = \frac{y^k\left(P_{m+k}(s, 1) + yP_{m+k+1}(s, 1) + \dots\right)}{Q_m(s, 1) + yQ_{m+1}(s, 1) + \dots}.$$

So, if t_0 (resp. s_0) is a root of $Q_m(1, t)$ (resp. $Q_m(s, 1)$), then the point $(x = 0, t = t_0)$ (resp. $(s = s_0, y = 0)$) would be an essential singularity of $\widetilde{M}$. The second eventuality, $n = m$ and P_n, Q_n are not coprime, also cannot occur. Indeed, $P_n(1, t)$ and $Q_n(1, t)$ would have a common root t_0, and then

$$\begin{aligned}\widetilde{M}(x, t) &= \frac{P_n(1, t) + xP_{n+1}(1, t) + \dots}{Q_n(1, t) + xQ_{n+1}(1, t) + \dots} \\ &= \frac{(t - t_0)P_n^*(1, t) + xP_{n+1}(1, t) + \dots}{(t - t_0)Q_n^*(1, t) + xQ_{n+1}(1, t) + \dots}\end{aligned}$$

would have an essential singularity at $(x = 0, t = t_0)$.

2. An intermediate result

We prove in this section the existence of a particular homeomorphism between two germs of analytic varieties that coincide to high order at $0 \in \mathbf{C}^2$. We will denote by $\mathbf{C}_c$ the complex line in $\mathbf{C}^2$ given by $y = cx$, where c is a complex number.

Proposition 2.1. *Let $f(x, y)$ be the germ at $0 \in \mathbf{C}^2$ of a holomorphic function and let $f_k(x, y)$ be its kth jet. Define*

$$V(f) = \{(x, y) ; f(x, y) = 0\} \quad \text{and} \quad V(f_k) = \{(x, y) ; f_k(x, y) = 0\} .$$

Then, for k large enough, there is a germ of a homeomorphism Φ at $0 \in \mathbf{C}^2$ such that

$$\Phi(V(f)) = V(f_k) \quad \text{and} \quad \Phi(\mathbf{C}_c) = \mathbf{C}_c$$

for every $c \in \mathbf{C}$.

The proof of this proposition is based upon some lemmas.

Lemma 2.1. *Consider a family $D(p_i, r_i)$, $i = 1, \dots, N$, of N disjoint discs in the plane $\mathbf{R}^2$, with respective centers p_i and radii r_i. For every i, let q_i be a point in $D(p_i, r_i)$. Then there is a homeomorphism $\phi\colon \mathbf{R}^2 \mapsto \mathbf{R}^2$ such that:*

$$\phi(p) = p \quad \text{if } p \notin \bigcup_{i=1}^{N} D(p_i, r_i)$$
$$\phi(p_i) = q_i \quad \text{for } i = 1, \dots, N.$$

We leave the elementary proof of this result to the reader.

Lemma 2.2. *Let $f(x, y)$ be the germ at $0 \in \mathbf{C}^2$ of an irreducible holomorphic function of order n. Then, we can assume, after a linear change of coordinates, that*

$$f(x, y) = y^n + P_{n+1}(x, y) + \dots + P_{n+k}(x, y) + \dots ,$$

where the P_{n+j}'s are homogeneous polynomials of degree $n + j$. Furthermore, if $n > 1$, then there is a k such that $P_{n+k}(x, 0)$ is not identically zero.

Proof. The Taylor expansion of f is $f(x, y) = \sum_{j \geq n} P_j(x, y)$, where the P_j's are homogeneous polynomials of degree j. Recall from [4] that the irreducibility of f implies that the tangent cone of the variety $\{(x, y); f(x, y) = 0\}$ is reduced to a point. That is, the set of points on $\mathbf{CP}^1$ on which $P_n(1, t)$ or $P_n(s, 1) = 0$ is reduced to one point. We can assume, after a linear change of coordinates if necessary, that $s = 0$ is not a root of $P_n(s, 1)$. So, the irreducibility of f implies that $P_n(1, t) = (t - t_0)^i$ for some $t_0 \in \mathbf{C}$ and $i \leq n$. By a linear change of coordinates, we can assume $t_0 = 0$. Now, if $n > i$, then $P_n(s, 1) = s^{n-i}$ would have a root at $s = 0$, and this is impossible. So $i = n$ and $P_n(x, y) = y^n$. Finally, if $n > 1$ and $P_{n+j}(x, 0) = 0$ for every

$j \geq 1$, then y would be a factor of f and f would be reducible. The lemma is proved. □

Lemma 2.3. *Let $f(x, y) = y^n + P_{n+1}(x, y) + \ldots + P_{n+k}(x, y) + \ldots$, be an irreducible holomorphic function and k be the first integer such that $P_{n+k}(x, 0) \neq 0$. Then there is a positive constant K such that, for every $c \in \mathbf{C}$, $0 < |c| < K$, the variety $V(f) = \{f = 0\}$ intersects the line $\mathbf{C}_c$ at 0 and k other points $p^1(c), \ldots, p^k(c)$ which vary continuously with c and $p^1(0) = \ldots = p^k(0) = 0$.*

Proof. Let c be fixed in $\mathbf{C}$. Then

$$V(f) \cap \mathbf{C}_c = \left\{(x, cx) \in \mathbf{C}^2 ; f(x, cx) = 0\right\},$$

i.e., $V(f) \cap \mathbf{C}_c$ is given by the equation

$$x^n \left(c^n + P_{n+1}(1, c)x + \ldots + P_{n+k}(1, c)x^k + \ldots\right) = 0.$$

Since $P_{n+j}(1, 0) = 0$ if $j < k$ and $P_{n+k}(1, 0) \neq 0$, then similar arguments as that used in the proof of Theorem 4.13 of Chapter 2 in [2] and the irreducibility of f show that the equation

$$c^n + P_{n+1}(1, c)x + \ldots + P_{n+k}(1, c)x^k + \ldots = 0$$

has k distinct and continuous roots $p^1(c), \ldots, p^k(c)$, provided that $c \neq 0$ is sufficiently small and $p^1(0) = \ldots = p^k(0) = 0$. □

Proof of the proposition. We start by proving the proposition in the case where the tangent cone of $V(f)$ is reduced to one point. Let $K > 0$ be such that, for every $c \in \mathbf{C}$, $0 < |c| < K$, we have

$$V(f) \cap \mathbf{C}_c = \left\{0, p^1(c), \ldots, p^k(c)\right\}$$

(see Lemma 2.3). Consider the homotopy

$$g_t = f + t\left(f_l - f\right), \qquad 0 \leq t \leq 1,$$

where f_l is the lth jet of f. Since f and g_t coincide to order l at 0, then, for l sufficiently large, we also have

$$V(g_t) \cap \mathbf{C}_c = \left\{0, p_t^1(c), \ldots, p_t^k(c)\right\},$$

and the functions $p_t^j(c)$'s are continuous in c and t. So, for every j, we can find a continuous function $r_j(c)$, $r_j(c) > 0$ and $r_j(0) = 0$ such that

$$\left\{p_t^j(c), 0 \leq t \leq 1\right\} \subset D\left(p^j(c), r_j(c)\right)$$

and

$$D\left(p^j(c), r_j(c)\right) \cap D\left(p^i(c), r_i(c)\right) = \varnothing \quad \text{if } j \neq i.$$

Next, for c fixed, $|c| < K$, consider, as in Lemma 2.1, a homeomorphism ϕ_c of the complex line $\mathbf{C}_c$ such that

$$\phi_c(p) = p \quad \text{if } p \in \mathbf{C}_c - \bigcup_{i=1}^{k} D\left(p^i(c), r_i(c)\right),$$
$$\phi_c\left(p^i(c)\right) = p_1^i(c) \quad \text{for } i = 1, \dots, k.$$

Note that if U is a small neighborhood of $0 \in \mathbf{C}^2$ and c is such that $K/2 < |c| < K$, then $\phi_c(p) = p$ if $p \in U \cap \mathbf{C}_c$. So, we can define a homeomorphism Φ: $(\mathbf{C}^2, 0) \mapsto (\mathbf{C}^2, 0)$ by

$$\Phi(p) = p \quad \text{if } p \notin \{(x, y) ; y = cx \text{ and } |c| < K\}$$
$$\Phi(p) = \phi_c(p) \quad \text{if } p \in \mathbf{C}_c \text{ and } |c| < K.$$

In this case Φ is the desired homeomorphism of the proposition.

In the general case, the tangent cone of $V(f)$ is not necessarily reduced to a point; we decompose

$$V(f) = V^1 \cup \dots \cup V^N,$$

where each V^j has a tangent cone reduced to a point a_j. Similarly, we decompose

$$V(f_k) = V_k^1 \cup \dots \cup V_k^N,$$

where the tangent cone of V_k^j is the point a_j. By reducing the neighborhood U of $0 \in \mathbf{C}^2$, we can assume that, for every j, there is a K_j such that $V^j \cap \mathbf{C}_c = \{0\}$ if $|c - a_j| \geq K_j$ and the discs $D(a_j, K_j)$ are mutually disjoint. Now we can define a homeomorphism Φ as follows:

$$\Phi(p) = p \quad \text{if } p \notin \bigcup_{j=1}^{N} \left\{(x, y) ; y = cx \text{ and } \left|c - a_j\right| \geq K\right\}$$
$$\Phi(p) = \Phi_j(p) \quad \text{if } p \in \left\{(x, y) ; y = cx \text{ and } \left|c - a_j\right| < K_j\right\},$$

where Φ_j is a homeomorphism, as in the previous case, corresponding to V^j. The proposition is therefore proved. □

3. Statement and proof of the main result

Theorem. *Let $M(x, y)$ be the germ at $0 \in \mathbf{C}^2$ of a meromorphic function in the class A. Then M is topologically conjugate to a rational function.*

The proof of this result needs two lemmas. First, we adopt the following notation: if $V(f)$ is the germ of the variety $\{f = 0\}$, we will denote by $\widetilde{V}(f)$ the closure in $\mathbf{C}_0^2$ of $E^{-1}(V(f) - \{0\})$. Also, we will say that two varieties $V(f)$ and $V(g)$ coincide to order k if their defining functions f and g coincide to order k at 0.

Lemma 3.1. *Let f and g be germs of irreducible holomorphic functions of order j at $0 \in \mathbf{C}^2$. Suppose that f and g coincide to order k $(k > j)$ at 0. Then $\widetilde{V}(f)$ and $\widetilde{V}(g)$ coincide to order $k - j$ on $E^{-1}(0)$.*

Proof. Without loss of generality, we can assume that the tangent cone of $V(f)$ or $V(g)$ is not the point $(s = 0,\ y = 0)$. So the varieties $\widetilde{V}(f)$ and $\widetilde{V}(g)$ admit, respectively, $\tilde{f}(x, t) = f(x, tx)/x^j$ and $\tilde{g}(x, t) = g(x, tx)/x^j$ as defining functions. Therefore, it follows from the fact that f and g coincide to order k at 0, that $\tilde{f}(x, t)$ and $\tilde{g}(x, t)$ coincide to order $k - j$ on $E^{-1}(0)$.

Lemma 3.2. *Suppose that A, B are germs of holomorphic functions that coincide to order k_1 and $V(f)$, $V(g)$ are germs of analytic varieties that coincide to order k_2 at $0 \in \mathbf{C}^2$. Then the images via the maps $F(x, y) = (x, A(x, y))$ and $G(x, y) = (x, B(x, y))$ of, respectively, $V(f)$ and $V(g)$ coincide to an order $\nu(k_1, k_2)$, where ν increases with k_1, k_2.*

Proof. Let p be the order of f and g at 0. Then a parametrization of $V(f)$ (resp. $V(g)$) is (see [4])

$$\left(x = t^p,\ y = a(t)\right) \qquad \left(\text{resp. } \left(x = t^p,\ y = b(t)\right)\right),$$

where a, b are holomorphic functions of order $\geq p + 1$. Moreover, it follows from the fact that f and g coincide to order k_2 that a and b coincide to an order $\eta(k_2)$, where η increases with k_2. The parametrization of $F(V(f))$ (resp. $G(V(g))$) is

$$\left(x = t^p,\ y = A\left(t^p, a(t)\right)\right) \qquad \left(\text{resp. } \left(x = t^p,\ y = B\left(t^p, b(t)\right)\right)\right).$$

The conclusion follows from the fact that A, B coincide to order k_1 and a, b coincide to order $\eta(k_2)$. □

Proof of the theorem. We write

$$M(x, y) = \frac{\alpha(x, y)}{\beta(x, y)} = \frac{\sum_{j \geq n} P_j(x, y)}{\sum_{j \geq n} Q_j(x, y)},$$

where P_j, Q_j are homogeneous polynomials of degree j and P_n, Q_n coprime. For $k > n$, define the rational function R_k by

$$R_k(x, y) = \frac{\alpha_k(x, y)}{\beta_k(x, y)} = \frac{\sum_{j=n}^{k} P_j(x, y)}{\sum_{j=n}^{k} Q_j(x, y)}.$$

Note that the blow-up $\widetilde{M}$ of M coincides on $E^{-1}(0)$ with the blow-up $\widetilde{R}_k$ of R_k, it is

$$\widetilde{M}(0, t) = \widetilde{R}_k(0, t) = \frac{P_n(1, t)}{Q_n(1, t)}.$$

Consider the maps $\mathscr{P}: \mathbf{C}_0^2 \mapsto \mathbf{C}_0^2$ and $\mathscr{P}_k: \mathbf{C}_0^2 \mapsto \mathbf{C}_0^2$, defined in the charts (x, t), (s, y) by

$$\mathscr{P}(x, t) = \left(x, \widetilde{M}(x, t)\right) \quad \text{and} \quad \mathscr{P}(s, y) = \left(\frac{1}{\widetilde{M}(sy, 1/s)}, sy\widetilde{M}\left(sy, \frac{1}{s}\right)\right),$$

$$\mathscr{P}_k(x, t) = \left(x, \widetilde{R}_k(x, t)\right) \quad \text{and} \quad \mathscr{P}_k(s, y) = \left(\frac{1}{\widetilde{R}_k(sy, 1/s)}, sy\widetilde{R}_k\left(sy, \frac{1}{s}\right)\right).$$

Note that the meromorphic map $\mathscr{P}$ (resp. $\mathscr{P}_k$) is holomorphic outside the variety $\{\widetilde{M} = \infty\}$ (resp. $\{\widetilde{R}_k = \infty\}$). Since the jacobian of $\mathscr{P}$ (resp. $\mathscr{P}_k$), in the chart (x, t), vanishes only on the variety

$$X = \left\{(x, t)\,;\, \frac{\partial \widetilde{M}}{\partial t} = 0\right\} \qquad \left(\text{resp. } X_k = \left\{(x, t)\,;\, \frac{\partial \widetilde{R}_k}{\partial t} = 0\right\}\right),$$

then $\mathscr{P}$ (resp. $\mathscr{P}_k$) is a covering outside X (resp. X_k) whose number of sheets is the multiplicity of $\widetilde{M}(0, t)$. Since

$$E(X) = \left\{(x, y)\,;\, \beta\frac{\partial \alpha}{\partial y} - \alpha\frac{\partial \beta}{\partial y} = 0\right\}$$

and

$$E(X_k) = \left\{(x, y)\,;\, \beta_k\frac{\partial \alpha_k}{\partial y} - \alpha_k\frac{\partial \beta_k}{\partial y} = 0\right\}$$

and the functions $\beta\frac{\partial \alpha}{\partial y} - \alpha\frac{\partial \beta}{\partial y}$, $\beta_k\frac{\partial \alpha_k}{\partial y} - \alpha_k\frac{\partial \beta_k}{\partial y}$ coincide to order $2k-1$, then X and X_k coincide to an order $\eta(k)$ (see Lemma 3.1). Therefore the varieties $W = \mathscr{P}(X)$ and $W_k = \mathscr{P}_k(X_k)$ coincide to a certain order $\mu(k)$, and μ increases with k (see Lemma 3.2).

Now, it follows from Proposition 2.1 that if k is large enough, then there is a homeomorphism $\Psi: (\mathbf{C}^2, 0) \mapsto (\mathbf{C}^2, 0)$ such that

$$\widetilde{\Psi}(W) = W_k \quad \text{and} \quad \widetilde{\Psi}(\{(x, t)\,;\, t = t_0\}) = \{(x, t)\,;\, t = t_0\},$$

for every constant t_0. Next, since $\mathscr{P}$ (resp. $\mathscr{P}_k$) maps the analytic varieties $\widetilde{M}^{-1}(\widetilde{M}(x_0, t_0))$ (resp. $\widetilde{R}_k^{-1}(\widetilde{R}_k(x_0, t_0))$) into the complex line $t = t_0$, there is a homeomorphism $\widetilde{\Phi}$ such that the diagram

$$\begin{array}{ccc} \left(\mathbf{C}_0^2, X\right) & \xrightarrow{\widetilde{\Phi}} & \left(\mathbf{C}_0^2, X_k\right) \\ \downarrow{\scriptstyle \mathscr{P}} & & \downarrow{\scriptstyle \mathscr{P}_k} \\ \left(\mathbf{C}_0^2, W\right) & \xrightarrow{\widetilde{\Psi}} & \left(\mathbf{C}_0^2, W_k\right) \end{array}$$

commutes, i.e., $\widetilde{\Phi}$ conjugates the foliation defined by the level sets of $\widetilde{M}$ to that defined by the level sets of $\widetilde{R}_k$. Since $\widetilde{M} = \widetilde{R}_k$ on $E^{-1}(0)$ and $\widetilde{\Psi}$ is the identity on $E^{-1}(0)$, then $\widetilde{\Phi}$ can be chosen to be the identity on $E^{-1}(0)$. Finally, the homeomorphism $\Phi = E \circ \widetilde{\Phi} \circ E^{-1}$, defined in a neighborhood of $0 \in \mathbf{C}^2$, conjugates R_k with M, i.e., $R_k \circ \Phi = M$. The theorem is proved. □

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