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TWO FAMILIES OF STABLE BUNDLES WITH THE SAME SPECTRUM

A. P. RAO

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ABSTRACT. We study stable rank two algebraic vector bundles on $\mathbf{P}^3$ and show that the family of bundles with fixed Chern classes and spectrum may have more than one irreducible component. We also produce a component where the generic bundle has a monad with ghost terms which cannot be deformed away.

In the last century, Halphen and M. Noether attempted to classify smooth algebraic curves in $\mathbf{P}^3$. They proceeded with the invariants d, g, the degree and genus, and worked their way up to $d = 20$. As the invariants grew larger, the number of components of the parameter space grew as well. Recent work of Gruson and Peskine has settled the question of which invariant pairs (d, g) are admissible. For a fixed pair (d, g), the question of determining the number of components of the parameter space is still intractable. For some values of (d, g) (for example [E-1], if $d \geq g + 3$) there is only one component. For other values, one may find many components including components which are nonreduced [M].

More recently, similar questions have been asked about algebraic vector bundles of rank two on $\mathbf{P}^3$. We will restrict our attention to stable bundles $\mathscr{E}$ with first Chern class $c_1 = 0$ (and second Chern class $c_2 = n$, so that n is a positive integer and $\mathscr{E}$ has no global sections). To study these, we have the invariants n, the α-invariant (which is equal to $\dim H^1(\mathbf{P}^3, \mathscr{E}(-2)) \bmod 2$) and the spectrum. Stable bundles with $c_1 = 0$ exist for all values of $n > 0$ and, if $n > 1$, for both values 0 and 1 of the α-invariant. It is known that, as n grows large, the number of components of the moduli space of stable bundles with second Chern class n also grows large [E-2].

The spectrum χ of a stable rank two bundle on $\mathbf{P}^3$ with Chern classes $(0, n)$ is a connected sequence of n integers which describes the growth of the dimensions $M_i = H^1(\mathbf{P}^3, \mathscr{E}(i))$ in the range $i \leq -1$ [B-E]. It is symmetric about 0 and is subject to further constraints [H-2]. For low values of n, one can, by

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looking at all the possible spectra, obtain effective descriptions of all possible bundles (see [H-R] for $n \leq 8$). The relationship between the spectrum and the components of the moduli space is not clear. A component may parametrize bundles with different spectra [H-R]. In this note, we show that the parameter space of stable bundles with Chern classes $(0, 16)$ and spectrum $(-3, -2^4, -1^2, 0^2, 1^2, 2^4, 3)$ has at least two irreducible components. We do not know if these components are also irreducible components in the moduli space of *all* stable bundles with these Chern classes. Nevertheless, this example seems to suggest that, as n grows large, we will require more invariants to distinguish components in the moduli space.

Suppose $\mathscr{E}$ is a stable rank two bundle on $\mathbf{P}^3$ (over an algebraically closed field of characteristic 0) with $c_1 = 0$, $c_2 = 16$ and spectrum $\chi = (-3, -2^4, -1^2, 0^2, 1^2, 2^4, 3)$. Let M be its cohomology module $H^1_*(\mathscr{E})$. Then if ε_i is the number of minimal generators for M in degree $-i$, we have, by [B],

$$\dim M_i = 1, 6, 13, 22 \quad \textit{for } i = -4, -3, -2, -1 \textit{ respectively},$$

and $\varepsilon_4 = 1$, $2 \leq \varepsilon_3 \leq 3$, $\varepsilon_2 \leq 1$, $\varepsilon_1 \leq 1$.

Definition. We say that $\mathscr{E}$ has type α if $\varepsilon_3 = 2$ and has type β if $\varepsilon_3 = 3$.

Theorem. *There is a maximal family $\mathscr{M}_1$ of bundles of type α and a maximal family $\mathscr{M}_2$ of bundles of type β, neither of which specializes to the other. Hence the parameter space of stable bundles with Chern classes* $(0, 16)$ *and spectrum* $(-3, -2^4, -1^2, 0^2, 1^2, 2^4, 3)$ *has at least two irreducible components.*

Proof. We construct below a bundle of type α and a bundle of type β. The bundle of type β will have $h^0(\mathscr{E}(2)) = 2$. In order to obtain the two families $\mathscr{M}_1$ and $\mathscr{M}_2$, choose maximal irreducible families of bundles with this spectrum containing the two constructed bundles.

Suppose now that T is some parameter scheme and that $\mathscr{F}$ is a rank two bundle on $\mathbf{P}^3 \times T$ whose restriction to each fibre of $\pi\colon \mathbf{P}^3 \times T \to T$ is stable with $c_1 = 0$ and spectrum χ. Let A be the kernel of the homomorphism

$$R^1\pi_*\mathscr{F}(-4) \bigotimes_{\mathscr{O}_T} H^0(\mathscr{O}_{\mathbf{P}}(1)) \to R^1\pi_*\mathscr{F}(-3).$$

Then the locus of bundles of type β in the family is the support of A on T and is therefore closed in T. Hence the general bundle in $\mathscr{M}_1$ has type α. On the other hand, we analyse below the minimal monads of bundles of type α and show that $h^0(\mathscr{E}(2)) > 2$ for any such bundle. Since cohomology module dimensions are upper semi-continuous on parameter spaces, the general bundle in $\mathscr{M}_2$ has type β. The same considerations show that $\mathscr{M}_1 \not\subset \overline{\mathscr{M}}_2$ and $\mathscr{M}_2 \not\subset \overline{\mathscr{M}}_1$.

Minimal monads for bundles of type α. The constraints on the number of minimal generators for M yield a minimal monad for $\mathscr{E}$ of the form

$$\mathscr{A} \xrightarrow{\varphi} \mathscr{B} \xrightarrow{\psi} \mathscr{C},$$

where

$$\mathscr{A} = \mathscr{C}^{\vee},$$
$$\mathscr{B} = (5+\varepsilon_2)\mathscr{O}_{\mathbf{P}}(-2) \oplus \gamma\mathscr{O}_{\mathbf{P}}(-1) \oplus (2+2\varepsilon_0)\mathscr{O}_{\mathbf{P}} \oplus \gamma\mathscr{O}_{\mathbf{P}}(1) \oplus (5+\varepsilon_2)\mathscr{O}_{\mathbf{P}}(2),$$
$$\mathscr{C} = \mathscr{O}_{\mathbf{P}}(4) \oplus 2\mathscr{O}_{\mathbf{P}}(3) \oplus \varepsilon_2\mathscr{O}_{\mathbf{P}}(2) \oplus \varepsilon_1\mathscr{O}_{\mathbf{P}}(1) \oplus \varepsilon_0\mathscr{O}_{\mathbf{P}} \oplus (2+\gamma-\varepsilon_1)\mathscr{O}_{\mathbf{P}}(-1),$$

γ is a nonnegative integer and ϕ and ψ are respectively injective and surjective maps of vector bundles.

We analyse this monad using three tools:

(1) If $\mathscr{F} \hookrightarrow \mathscr{G}$ is a non-split injection as vector bundles of free sheaves on $\mathbf{P}^3$, then $\operatorname{rank}(\mathscr{F}) \leq \operatorname{rank}(\mathscr{G}) - 3$. (To see this, notice that the cokernel of the injection is a vector bundle on $\mathbf{P}^3$ for which, since the injection does not split, H^2_* is non-zero. Hence it is not a line bundle. Since it has $H^1_* = 0$, by Serre duality it cannot have rank two either.)

(2) Let $\mathscr{F}$ denote successively the cokernels of the following injections of vector bundles on $\mathbf{P}^3$:

(i) $2\mathscr{O}_{\mathbf{P}}(1) \hookrightarrow 5\mathscr{O}_{\mathbf{P}}(2)$
(ii) $3\mathscr{O}_{\mathbf{P}}(1) \hookrightarrow 6\mathscr{O}_{\mathbf{P}}(2)$
(iii) $\mathscr{O}_{\mathbf{P}} \oplus \mathscr{O}_{\mathbf{P}}(1) \hookrightarrow 5\mathscr{O}_{\mathbf{P}}(2)$
(iv) $2\mathscr{O}_{\mathbf{P}} \oplus \mathscr{O}_{\mathbf{P}}(1) \hookrightarrow 6\mathscr{O}_{\mathbf{P}}(2)$

In cases (i) and (ii), $H^0(\mathscr{F}^{\vee}(3)) = 0$. In cases (iii) and (iv), $H^0(\mathscr{F}^{\vee}(n)) = 0$ for $n = 3, 4$.

(To see this, one constructs an appropriate smooth curve. Consider (i) for example. Two general sections of $\mathscr{F}(-2)$ have for their dependency locus a twisted cubic curve, C, giving the exact sequence

$$0 \to \mathscr{O}_{\mathbf{P}}(-4) \to \mathscr{F}^{\vee} \to 2\mathscr{O}_{\mathbf{P}}(-2) \xrightarrow{\delta} \omega_{\mathbf{C}} \to 0.$$

Hence $H^0(\mathscr{F}^{\vee}(3)) = H^0(\mathscr{K}(3))$ where $\mathscr{K} = \ker\delta$. But δ factors through $2\mathscr{O}_C(-2)$, and we get the result from the sequence

$$0 \to 2\mathscr{I}_C(-2) \to \mathscr{K} \to \omega_C^{\vee}(-4) \to 0.)$$

(3) Let Q be a rank r subsheaf of $\mathscr{O}_{\mathbf{P}}(a_1) \oplus \cdots \oplus \mathscr{O}_{\mathbf{P}}(a_s)$ where $a_1 \geq a_2 \geq \cdots \geq a_s$. Then $h^0(Q) \leq h^0(\mathscr{O}_{\mathbf{P}}(a_1) \oplus \cdots \oplus \mathscr{O}_{\mathbf{P}}(a_r))$ (as we see from a general projection onto the first r summands.)

Using these tools in a case by case analysis of the monad, we show that, for any such bundle $\mathscr{E}$, $h^0(\mathscr{E}(2)) > 2$. ([H-R] contains details of similar analyses.) We will demonstrate a typical argument establishing one case:

$\varepsilon_2 = 0$ or 1. Suppose $\varepsilon_2 = 0$. We can extract two subcomplexes from the minimal monad:

$$(2+\gamma-\varepsilon_1)\mathscr{O}_{\mathbf{P}}(1) \hookrightarrow 5\mathscr{O}_{\mathbf{P}}(2) \to \mathscr{O}_{\mathbf{P}}(4) \oplus 2\mathscr{O}_{\mathbf{P}}(3)$$
$$\varepsilon_0\mathscr{O}_{\mathbf{P}} \oplus (2+\gamma-\varepsilon_1)\mathscr{O}_{\mathbf{P}}(1) \hookrightarrow \gamma\mathscr{O}_{\mathbf{P}}(1) \oplus 5\mathscr{O}_{\mathbf{P}}(2) \to \mathscr{O}_{\mathbf{P}}(4) \oplus 2\mathscr{O}_{\mathbf{P}}(3)$$

(where the left hand maps are injections of vector bundles).

Applying (1) to the left hand map of the first subcomplex, we see that $2+\gamma-\varepsilon_1 \leq 2$. We will deal here with the case $2+\gamma-\varepsilon_1 = 2$. Then $\gamma = \varepsilon_1 \leq 1$. Applying (1) to the left hand map of the second subcomplex, we see that $\varepsilon_0 \leq \gamma$, so $\varepsilon_0 \leq 1$. Now using (2)(i) on the cokernel $\mathscr{F}$ of the injection in the first subcomplex, we see that $\phi\colon \mathscr{F} \to \mathscr{O}_{\mathbf{P}}(4) \oplus 2\mathscr{O}_{\mathbf{P}}(3)$ has image in $\mathscr{O}_{\mathbf{P}}(4)$. Hence we calculate that $h^0(\ker\phi(1)) \geq 24$ (we essentially use (3) here.) Now if a section of $\ker\phi(1)$ is not in the image of $H^0([\mathscr{A}/2\mathscr{O}_{\mathbf{P}}(1)](1))$ (which has dimension $\varepsilon_1+4\varepsilon_0$), it actually descends to a non-zero section of $\mathscr{E}(1)$. Since $\varepsilon_1+4\varepsilon_0 \leq 5$, we conclude that $h^0(\mathscr{E}(1)) \geq 19$ (of course we are not claiming that there exists a stable bundle with this particular monad form.) Hence in this case, if such an $\mathscr{E}$ exists, then $h^0(\mathscr{E}(2)) > 2$.

Construction of a bundle of type α. Fix six points $p_1, \ldots, p_6$ in general position on $\mathbf{P}^2$ and find a plane curve of degree 7 passing through p_1 whose only singularities are nodes at p_2, p_3, p_4 and triple points at p_5, p_6. (Its existence follows from [H-1, Cor. V.4.13.]) When a smooth cubic surface in $\mathbf{P}^3$ is realized as the blow up of the plane at these points, the proper transform of the plane curve is a smooth space curve X of degree 8 and genus 6 with the properties:

$H^0_*(\omega_X)$ has exactly 3 minimal generators, 1 in degree -1 and 2 in degree 0;

$H^1_*(\mathscr{I}_X)$ is a one dimensional vector space supported in degree 2.

Choose a nowhere vanishing section of $\mathscr{N}_X \otimes \omega_X(2)$. We get a curve Y' and a stable rank two bundle $\mathscr{E}'$ with $c_1 = 0$ and $c_2 = 15$, satisfying

$$0 \to \mathscr{I}_{Y'} \to \mathscr{I}_X \to \omega_X(2) \to 0$$
$$0 \to \mathscr{O}_{\mathbf{P}}(-1) \to \mathscr{E}' \to \mathscr{I}_{Y'}(1) \to 0.$$

If $M' = H^1_*(\mathscr{E}')$, then M' has 4 minimal generators: one in degree -4, two in degree -3 (all obtained from $H^0_*(\omega_X)$) and one in degree 1 (obtained from $H^1_*(\mathscr{I}_X)$), and

$$\dim M'_i = 1, 6, 13, 21, \quad \textit{for } i = -4, -3, -2, -1 \textit{ respectively.}$$

So $\mathscr{E}'$ has spectrum $(-3, -2^4, -1^2, 0, 1^2, 2^4, 3)$ and minimal monad

$$0 \to \mathscr{O}_{\mathbf{P}}(1) \oplus 2\mathscr{O}_{\mathbf{P}}(-3) \oplus \mathscr{O}_{\mathbf{P}}(-4) \to 5\mathscr{O}_{\mathbf{P}}(-2) \oplus 5\mathscr{O}_{\mathbf{P}}(2) \to \mathscr{O}_{\mathbf{P}}(4) \oplus 2\mathscr{O}_{\mathbf{P}}(3) \oplus \mathscr{O}_{\mathbf{P}}(-1) \to 0.$$

Pick a general line L disjoint from Y'. Then $Y = Y' \cup L$ is a curve which is the zero scheme of a section of $\mathscr{E}(1)$ where $\mathscr{E}$ is a stable rank two bundle with $c_1 = 0$, $c_2 = 16$, spectrum $(-3, -2^4, -1^2, 0^2, 1^2, 2^4, 3)$ and monad

$$0 \to \mathscr{O}_{\mathbf{P}}(1) \oplus \mathscr{O}_{\mathbf{P}}(-1) \oplus 2\mathscr{O}_{\mathbf{P}}(-3) \oplus \mathscr{O}_{\mathbf{P}}(-4) \to 5\mathscr{O}_{\mathbf{P}}(-2) \oplus 2\mathscr{O}_{\mathbf{P}} \oplus 5\mathscr{O}_{\mathbf{P}}(2) \to \mathscr{O}_{\mathbf{P}}(4) \oplus 2\mathscr{O}_{\mathbf{P}}(3) \oplus \mathscr{O}_{\mathbf{P}}(1) \oplus \mathscr{O}_{\mathbf{P}}(-1) \to 0.$$

This is the bundle of type α.

Construction of a bundle of type β with $h^0(\mathscr{E}(2)) = 2$. Fix a plane $H = 0$ in $\mathbf{P}^3$ and choose a smooth plane quintic P and a divisor D of three distinct points on P such that $H^0(\mathscr{O}_P(2-2D)) = 0$. Choose a general nowhere vanishing section of $\mathscr{N}_P \otimes \mathscr{O}_P(D-1)$. This gives a double structure X on P of degree 10 and arithmetic genus 13. H^2 belongs to $I(X)$ and locally gives a minimal generator of $\mathscr{I}_{X,x}$ at points x not in D. In fact, for a general choice of the section, $I(X)$ contains only multiples of H^2 in degrees ≤ 4.

We calculate that $H^1(\mathscr{N}_X \otimes \omega_X(-1)) = 0$, and hence a general choice of a section s of $\mathscr{N}_X \otimes \omega_X$ is nowhere vanishing. This gives rise to a double structure Y on X and (since the obstruction in Ferrand's theorem is zero here) a rank two bundle $\mathscr{E}$ with $c_1 = 0$ and $c_2 = 16$ where

$$0 \to \mathscr{I}_Y \to \mathscr{I}_X \to \omega_X \to 0,$$
$$0 \to \mathscr{O}_{\mathbf{P}}(-2) \to \mathscr{E} \to \mathscr{I}_Y(2) \to 0.$$

H^4 belongs to $I(Y)$ and we check that if s is sufficiently general, this is the only surface of degree 4 containing Y. Hence $h^0(\mathscr{E}(2)) = 2$ and, in particular, $\mathscr{E}$ is stable. To find its spectrum, observe that

$$H^1(\mathscr{E}(-4)) = H^1(\mathscr{I}_Y(-2) = H^0(\omega_X(-2))$$
$$H^1(\mathscr{E}(-3)) = H^1(\mathscr{I}_Y(-1) = H^0(\omega_X(-1)),$$

as $h^1(\mathscr{I}_X(-i)) = 0$ for $i \geq 0$.

Since we have

$$0 \to \omega_P \to \omega_X \to \omega_P(1-D) \to 0$$

and $\omega_P = \mathscr{O}_P(2)$, we get $h^0(\omega_X(-2)) = 1$ and $h^0(\omega_X(-1)) = 6$. This forces the spectrum to be χ. Furthermore, the contribution to $H^0(\omega_X(-2))$ from $\mathscr{O}_P$ is annihilated by the linear form H. Hence $\varepsilon_3 = 3$, and we have a bundle of type β.

Remark. A construction using the curve X of *Construction* β and a section s of $\mathscr{N}_X \otimes \omega_X(2)$ similar to the one above gives a stable bundle $\mathscr{E}$ with $c_1 = 0$, $c_2 = 19$ and spectrum $(-4, -3^4, -2^2, -1^2, 0, 1^2, 2^2, 3^4, 4)$ (see [H-R]). For this spectrum we must have $\varepsilon_5 = 1$ and $2 \leq \varepsilon_4 \leq 3$. The constructed bundle $\mathscr{E}$ has $\varepsilon_4 = 3$, and we can show by a similar analysis of the monads for this spectrum (using the three tools mentioned earlier) that there are no stable bundles with this spectrum for which $\varepsilon_4 = 2$. In this case the monad for $\mathscr{E}$ has the term $\mathscr{O}_{\mathbf{P}}(4)$ appearing in both the end term and the middle term of the minimal monad for $\mathscr{E}$. Such repetitions will be called "ghost terms" in the minimal monad since they make no contribution to the Hilbert function of the bundle. One might have guessed that such ghost terms should permit themselves to be deformed away. If this were true, the description of bundles by means of their monads would be easier since one could identify the more

generic monads as the ones without ghost terms. M. C. Chang shows in [C] that this happens in one case when $c_2 = 4$. The example of the body of this paper shows that this guess is not always true since bundles of type β have monads with ghost terms. The example of this remark shows that only monads with ghost terms may exist for some spectra.

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Department of Mathematics, University of Missouri, St. Louis, Missouri 63121