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A LOCAL BIFURCATION THEOREM FOR C^1-FREDHOLM MAPS

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ABSTRACT. The Krasnosel'skii Bifurcation Theorem is generalized to C^1-Fredholm maps. Let X and Y be Banach spaces, $F: R \times X \to Y$ be C^1-Fredholm of index 1 and $F(\lambda, 0) \equiv 0$. If $I \subseteq R$ is a closed, bounded interval at whose endpoints $\frac{\partial F}{\partial x}\frac{\partial F}{\partial x}(\lambda, 0)$ is invertible, and the parity of $\frac{\partial F}{\partial x}(\lambda, 0)$ on I is -1, then I contains a bifurcation point of the equation $F(\lambda, x) = 0$. At isolated potential bifurcation points, this sufficient condition for bifurcation is also necessary.

The celebrated Krasnosel'skii Local Bifurcation Theorem ([Kr]) asserts that if X is a Banach space and $C: X \to X$ is compact and differentiable at 0, with $C(0) = 0$, then each characteristic value of $C'(0)$ of odd algebraic multiplicity is a bifurcation point of

$$x - \lambda C(x) = 0.$$

Our purpose in this note is to present a generalization, based on the concept of parity introduced in [$F-P1$], of the Krasnosel'skii Theorem to one-parameter families of C^1-Fredholm maps. The proof is short, simple and uses only the classical change of degree argument. Moreover, at isolated potential bifurcation points, our sufficient condition for bifurcation is also necessary.

A number of extensions of the Krasnosel'skii Theorem have been given in the context of generalized multiplicities (see [M, S, I1, I2, L-M, C-H, Ki, E-L, E, Ra], among others). Certain technical aspects of the definitions of these multiplicities impose material restrictions on the class of maps for which bifurcation theorems can be proved.

As is standard, given Banach spaces X and Y, by $L(X, Y)$, $\Phi_n(X, Y)$, $\mathrm{GL}(X, Y)$ and $\mathscr{K}(X, Y)$ we denote the space of all bounded linear operators, the space of all linear Fredholm operators of Fredholm index n, the space of all linear isomorphisms, and the space of linear compact operators, respectively, from X to Y, endowed with the norm topology.

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Given a continuous path $\alpha\colon [a, b] = I \to \Phi_0(X, Y)$, a parametrix for α is a continuous path $\beta\colon I \to \mathrm{GL}(Y, X)$ such that

$$\beta(\lambda)\alpha(\lambda) - \mathrm{Id} \in \mathscr{K}(X, X) \qquad \text{for all } \lambda \in I.$$

Parametrices always exist (see [F-P1]).

If $\alpha\colon I \to \Phi_0(X, Y)$ is continuous and $\alpha(a)$ and $\alpha(b)$ are isomorphisms, then the parity of α on I is the element $\sigma(\alpha, I) \in \{+1, -1\}$ defined by

$$\sigma(\alpha, I) = \deg_{\mathrm{L.S.}}(\beta(a)\alpha(a)) \deg_{\mathrm{L.S.}}(\beta(b)\alpha(b)),$$

where $\beta\colon I \to \mathrm{GL}(Y, X)$ is any parametrix for α. Here, $\deg_{\mathrm{L.S.}}(\mathrm{Id}+K)$ is the Leray-Schauder degree with respect to any open set containing 0, when $\mathrm{Id}+K \in \mathrm{GL}(X, X)$ and $K \in \mathscr{K}(X, X)$ (see [L]).

That the parity is properly defined is proved in [F-P1, F-P3], in which one also finds a discussion of the properties of the parity. For specific calculations and comparisons with various concepts of generalized multiplicity, see [F-P2].

In the very special case in which each $\alpha(\lambda)$ is a compact vector field, $\sigma(\alpha, I)$ depends only on $\alpha(a)$ and $\alpha(b)$; in fact, one has $\sigma(\alpha, I) = -1$ if and only if $\deg_{\mathrm{L.S.}}(\alpha(a)) \neq \deg_{\mathrm{L.S.}}(\alpha(b))$. In general, this is not the case. One can have $\alpha(a) = \alpha(b)$ and $\sigma(\alpha, I) = -1$.

A C^1-Fredholm map $F\colon Z \to Y$ of index n is a continuously differentiable map such that the Fréchet derivative at any point is a linear Fredholm map of index n. If $F\colon \mathbf{R} \times X \to Y$ is C^1-Fredholm of index 1, then $\partial F/\partial x(\lambda, x)$ is a Fredholm operator of index 0 for all $(\lambda, x) \in \mathbf{R} \times X$.

Theorem 1. *Let $F\colon \mathbf{R} \times X \to Y$ be a C^1-Fredholm map of index 1 with $F(\lambda, 0) = 0$ for all $\lambda \in \mathbf{R}$, and let $L\colon \mathbf{R} \to \Phi_0(X, Y)$ be given by $L(\lambda) = \partial F/\partial x(\lambda, 0)$. Assume that for some $a < b$, $L(a)$ and $L(b)$ are invertible and*

$$\sigma(L, [a, b]) = -1.$$

Then every neighborhood of $[a, b] \times \{0\}$ contains solutions of

$$F(\lambda, x) = 0, \qquad \lambda \in \mathbf{R}, x \neq 0.$$

We recall that $\lambda_* \in \mathbf{R}$ is called a bifurcation point of $F(\lambda, x) = 0$ if each neighborhood of $(\lambda_*, 0)$ contains solutions of $F(\lambda, x) = 0$ with $x \neq 0$. So the above theorem asserts that (a, b) contains some bifurcation point.

According to the implicit function theorem, the set of bifurcation points is contained in

$$\Sigma = \{\lambda \in \mathbf{R} | L(\lambda) \notin \mathrm{GL}(X, Y)\}.$$

In linearized bifurcation theory, the goal is to obtain invariants, depending only on the family of linearizations $L(\lambda)$, which determine whether a point $\lambda \in \Sigma$ is actually a bifurcation point. Our next theorem shows that, at least at isolated points of Σ, the parity provides a complete invariant for linearized bifurcation theory.

Let $L: \mathbf{R} \to \Phi_0(X, Y)$ be continuous. We call $\lambda_* \in \Sigma$ an isolated singular point of L if λ_* is isolated in Σ. In this situation, we define the parity of L at λ_* by

$$\sigma(L, \lambda_*) = \lim_{\varepsilon \to 0} \sigma(L, [\lambda_* - \varepsilon, \lambda_* + \varepsilon]).$$

The right-hand limit exists (see [F-P1]).

From Theorem 1 it follows easily that if $\sigma(L, \lambda_*) = -1$, then λ_* is a bifurcation point for any C^1-Fredholm map F defined on a neighborhood of $(\lambda_*, 0)$ with $F(\lambda, 0) = 0$ and whose family of linearizations $\partial F/\partial x(\lambda, 0)$ coincides with $L(\lambda)$ near λ_*. On the other hand, we have

Theorem 2. *Let $L: \mathbf{R} \to \Phi_0(X, Y)$ be C^1 and $\lambda_* \in \mathbf{R}$ be an isolated singular point of L. Assume that*

$$\sigma(L, \lambda_*) = +1.$$

Then there is a neighborhood $\mathscr{O}$ of $(\lambda_*, 0)$ in $\mathbf{R} \times X$ and a C^1-Fredholm map $F: \mathscr{O} \to Y$ with

$$\frac{\partial F}{\partial x}(\lambda, 0) = L(\lambda) \qquad \text{for } (\lambda, 0) \in \mathscr{O}$$

and, if $(\lambda, x) \in \mathscr{O}$, then

$$F(\lambda, x) = 0 \qquad \text{if and only if } x = 0.$$

We now turn to the proofs of Theorems 1 and 2.

Proof of Theorem 1. Let $\beta: [a, b] = I \to \Phi_0(X, Y)$ be a parametrix for $\partial F/\partial x(\lambda, 0): I \to \Phi_0(X, Y)$. Then, of course, if $(\lambda, x) \in I \times X$

$$F(\lambda, x) = 0 \qquad \text{if and only if } \beta(\lambda)F(\lambda, x) = 0.$$

Accordingly, we set

$$\beta(\lambda)\frac{\partial F}{\partial x}(\lambda, 0) = \mathrm{Id} + K(\lambda)$$

and

$$\beta(\lambda)\left[F(\lambda, x) - \frac{\partial F}{\partial x}(\lambda, 0)\right] = R(\lambda, x),$$

so that

$$\beta(\lambda)F(\lambda, x) = x + K(\lambda)x + R(\lambda, x) \qquad \text{if } (\lambda, x) \in I \times X.$$

By definition, $K(\lambda) \in \mathscr{K}(X)$ if $\lambda \in I$, and, since $\sigma(\partial F/\partial x(\lambda, 0), I) = -1$,

$$\deg_{\mathrm{L.S.}}(\mathrm{Id} + K(a)) \neq \deg_{\mathrm{L.S.}}(\mathrm{Id} + K(b)). \tag{1}$$

Now, since F is C^1, it follows from the definition of R that $\partial R/\partial x(\lambda, x)$ exists and depends continuously on (λ, x) for $(\lambda, x) \in I \times X$. Clearly, $\partial R/\partial x(\lambda, 0) = 0$ and $R(\lambda, 0) = 0$. Choose $r > 0$ so that

$$\left\|\frac{\partial R}{\partial x}(\lambda, x)\right\| \leq \frac{1}{2} \qquad \text{if } (\lambda, x) \in I \times \overline{B}(0, r).$$

This, in turn, implies that $\mathrm{Id}+\partial R/\partial x(\lambda, x)$ is invertible if $(\lambda, x) \in I \times B(0, r)$, and that

$$\|R(\lambda, u) - R(\lambda, v)\| \leq \frac{1}{2}\|u - v\| \qquad \text{if } \lambda \in I;\ u, v \in B(0, r). \tag{2}$$

Fix $\lambda \in I$. We may apply the inverse function theorem to

$$x \mapsto x + R(\lambda, x), \qquad x \in B(0, r),$$

and so, letting $\delta = r/4$, obtain a function

$$g(\lambda, \cdot)\colon B(0, \delta) \to B(0, r)$$

which is C^1 and such that

$$g(\lambda, z) + R(\lambda, g(\lambda, z)) = z \qquad \text{if } z \in B(0, \delta). \tag{3}$$

From (3) and the fact that $\partial R/\partial x(\lambda, 0) = 0$ it follows that

$$\partial g/\partial z(\lambda, 0) = \mathrm{Id} \qquad \text{for all } \lambda \in I. \tag{4}$$

We use (2) and (3) to conclude that

$$\|u - v\| \geq \frac{1}{2}\|g(\lambda, u) - g(\lambda, v)\| \qquad \text{if } \lambda \in I;\ u, v \in B(0, r). \tag{5}$$

The continuity of $g\colon I \times B(0, \delta) \to X$ follows from (5).

Using (3), we see that if $(\lambda, z) \in I \times B(0, \delta)$, then

$$z + K(\lambda)g(\lambda, z) = 0 \qquad \text{if and only if } F(\lambda, g(\lambda, z)) = 0. \tag{6}$$

Accordingly, we set $C(\lambda, z) = K(\lambda)g(\lambda, z)$ and set

$$H(\lambda, z) = z + C(\lambda, z) \qquad \text{if } (\lambda, z) \in I \times B(0, \delta).$$

Then $C\colon I \times B(0, \delta) \to X$ is compact, and, in view of (4),

$$\frac{\partial H}{\partial z}(\lambda, 0) = \mathrm{Id} + K(\lambda) \qquad \text{for all } \lambda \in I.$$

Hence, from (1) we conclude that

$$\deg_{\mathrm{L.S.}}\left(\frac{\partial H}{\partial z}(a, 0)\right) \neq \deg_{\mathrm{L.S.}}\left(\frac{\partial H}{\partial z}(b, 0)\right).$$

We now follow Krasnosel'skii's argument and apply the homotopy invariance of the Leray-Schauder degree, together with the fact that

$$\deg_{\mathrm{L.S.}}(H(\lambda, \cdot), B(0, \varepsilon), 0) = \deg_{\mathrm{L.S.}}(\mathrm{Id} + K(\lambda))$$
$$\text{if } \lambda \in \{a, b\} \text{ and } \varepsilon > 0 \text{ is sufficiently small}$$

to conclude that there is some $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then there exist solutions of

$$z + K(\lambda)g(\lambda, z) = 0, \qquad \lambda \in I,\ \|z\| = \varepsilon.$$

Let $(\lambda_\varepsilon, z_\varepsilon)$ denote such a solution. Then, according to (6), $(\lambda_\varepsilon, g(\lambda_\varepsilon, z_\varepsilon))$ is a solution of $F(\lambda, x) = 0$. From (3) it follows that $g(\lambda_\varepsilon, z_\varepsilon) \neq 0$, while from (5) and the fact that $g(\lambda_\varepsilon, 0) = 0$ it follows that $\{g(\lambda_\varepsilon, z_\varepsilon\} \to 0$ as $\varepsilon \to 0$. □

The proof of Theorem 2 will rely on the invariance of the local parity under Lyapunov-Schmidt reduction ([F-P2, Theorem 4.10]) and a corresponding result, in the finite-dimensional case, of Ize ([I2, Theorem A]).

Proof of Theorem 2. Let $P \in \mathscr{L}(X)$ and $Q \in \mathscr{L}(Y)$ be projections onto $\operatorname{Ker} L(\lambda_*)$ and Range $L(\lambda_*)$, respectively. With respect to this choice of projections, let $S \in L(Y, X)$ be a pseudo-inverse of $L(\lambda_*)$, i.e.,

$$L(\lambda_*)SQ = Q \quad \text{and} \quad (I-P)S(I-Q) = 0. \tag{7}$$

Since $\mathrm{GL}(X)$ is open and L is continuous, we may choose a neighborhood I of λ_* so that

$$I + SQ(L(\lambda) - L(\lambda_*)) \in \mathrm{GL}(X) \qquad \text{if } \lambda \in I. \tag{8}$$

Moreover, since λ_* is an isolated singular point of $L(\lambda)$, we may also suppose that

$$L(\lambda) \in \mathrm{GL}(X, Y) \qquad \text{if } \lambda \in I \backslash \{\lambda_*\}. \tag{9}$$

Set

$$\widetilde{L}(\lambda) = L(\lambda)(I + SQ(L(\lambda) - L(\lambda_*)))^{-1} \qquad \text{if } \lambda \in I.$$

Then, with respect to the decompositions

$$X = P(X) \oplus (I-P)(X) \quad \text{and} \quad Y = (I-Q)(Y) \oplus Q(Y),$$

each $\widetilde{L}(\lambda)$ is represented as a 2 by 2 matrix of operators

$$\begin{bmatrix} l(\lambda) & A(\lambda) \\ B(\lambda) & M(\lambda) \end{bmatrix}.$$

From (7), (8) and (9) it follows that

$$B(\lambda) = 0 \quad \text{and} \quad M(\lambda) \in \mathrm{GL}((I-P)(X), Q(Y)) \qquad \text{if } \lambda \in I, \tag{10}$$

and that

$$l\colon I \to \mathscr{L}(P(X), (I-Q)(Y)) \text{ is } C^1 \text{ and invertible except at } \lambda = \lambda_*.$$

Fix bases in $P(X)$ and $(I-Q)(Y)$, and if $T \in \mathrm{GL}(P(X), (I-Q)(Y))$ denote by $\det T$ the determinant of the matrix representing T with respect to these bases.

According to [F-P2, Theorem 4.10],

$$\sigma(L, I) = \sigma(l, I) = \operatorname{sgn}\det[l(\lambda_* + \varepsilon)l(\lambda_* - \varepsilon)], \tag{11}$$

and so, since $\sigma(L, \lambda_*) = -1$, it follows from (11) that

$$\operatorname{sgn}\det[l(\lambda_* + \varepsilon)l(\lambda_* - \varepsilon)] = -1 \qquad \text{if } \varepsilon > 0 \text{ and } [\lambda_* - \varepsilon, \lambda_* + \varepsilon] \subseteq I.$$

According to [I2, Theorem A] (when k=1), there is a neighborhood U of $(\lambda_*, 0)$ in $I \times \operatorname{Ker} L(\lambda_*)$ and a C^1 mapping $\widetilde{R}\colon U \to \operatorname{Ker} Q$ with

$$\widetilde{R}(\lambda, 0) = 0 \quad \text{and} \quad \frac{\partial \widetilde{R}}{\partial x}(\lambda, 0) = 0 \qquad \text{if } (\lambda, 0) \in U, \tag{12}$$

and if $(\lambda, u) \in U$, then

$$l(\lambda)u + \widetilde{R}(\lambda, u) = 0 \qquad \text{if and only if } u = 0.$$

In view of (10), it follows that if $(\lambda, Pu) \in U$, then

$$(13)\quad L(\lambda)(I + KQ(L(\lambda) - L(\lambda_*)))^{-1}u + \widetilde{R}(\lambda, Pu) = 0 \qquad \text{if and only if } u = 0.$$

Finally, choose $\mathscr{O}$ a neighborhood of $(\lambda_*, 0)$ in $I \times X$ with

$$(\lambda, P(I + SQ(L(\lambda) - L(\lambda_*)))x) \in U \text{ if } \qquad (\lambda, x) \in \mathscr{O},$$

and define $F: \mathscr{O} \to Y$ by

$$F(\lambda, x) = L(\lambda)x + \widetilde{R}(\lambda, P(I + SQ(L(\lambda) - L(\lambda_*)))x) \qquad \text{if } (\lambda, x) \in \mathscr{O}.$$

Then $F: \mathscr{O} \to Y$ is C^1, from (12) it follows that

$$\frac{\partial F}{\partial x}(\lambda, 0) = L(\lambda) \qquad \text{if } (\lambda, 0) \in \mathscr{O},$$

and from (13) we conclude that if $(\lambda, x) \in \mathscr{O}$, then

$$F(\lambda, x) = 0 \qquad \text{if and only if } x = 0. \quad \square$$

It is clear that, in Theorems 1 and 2, F need only be defined on a neighborhood of $[a, b] \times \{0\}$ or of $(\lambda_*, 0)$, respectively. Moreover, if F is C^1 and $\partial F/\partial x(\lambda, 0) \in \Phi_0(X, Y)$, then F is C^1-Fredholm of index 1 in a neighborhood of $(\lambda, 0)$.

Corollary of Theorem 1. *Let $F: \mathbf{R} \times X \to Y$ be C^1, $\mathbf{R} \times \{0\} \subseteq F^{-1}(0)$ and $L: \mathbf{R} \to L(X, Y)$ be given by $L(\lambda) = \partial F/\partial x(\lambda, 0)$. Let $\lambda_* \in \mathbf{R}$ with $L(\lambda_*) \in \Phi_0(X, Y)$ and such that $L'(\lambda_*) = \partial^2 F/\partial\lambda\partial x(\lambda_*, 0)$ exists. Then every neighborhood of $(\lambda_*, 0)$ contains solutions of*

$$F(\lambda, x) = 0, \qquad \lambda \in \mathbf{R}, x \neq 0,$$

provided that one of the following two assumptions holds:

$$\text{(T)} \qquad \begin{cases} L'(\lambda_*)[\operatorname{Ker} L(\lambda_*)] \cap \operatorname{Range} L(\lambda_*) = \{0\}, \text{ and} \\ \dim \operatorname{Ker} L(\lambda_*) \text{ is odd}. \end{cases}$$

$$\text{(C)} \qquad \begin{cases} X = Y \text{ and } L(\lambda)L(\lambda_*) = L(\lambda_*)L(\lambda) \text{ for all } \lambda \in \mathbf{R}, \\ \operatorname{Ker} L'(\lambda_*) \cap \operatorname{Ker} L(\lambda_*) = \{0\}, \text{ and} \\ \dim \bigcup_{n=1}^{\infty} \operatorname{Ker}[L(\lambda_*)^n] \text{ is finite and odd}. \end{cases}$$

Proof. Since F is C^1, we may choose a neighborhood of $(\lambda_*, 0)$ on which F is C^1-Fredholm and $\partial F/\partial x(\lambda, 0) \in \Phi_0(X, Y)$. According to [F-P1, Theorems 5.2 and 5.21], (T) or (C) imply that λ_* is an isolated singular point of L and $\sigma(L, \lambda_*) = -1$. Let $[\lambda_* - \varepsilon, \lambda_* + \varepsilon)$ be an isolating neighborhood of λ_* in Σ. We invoke Theorem 1 with $[a, b] = [\lambda_* - \varepsilon, \lambda_* + \varepsilon]$. The conclusion follows by noting that λ_* is the only possible bifurcation point in $[\lambda_* - \varepsilon, \lambda_* + \varepsilon]$. $\square$

The commutativity assumption (C) has its origin in the original Krasnosel'skii Theorem in which

$$\frac{\partial F}{\partial x}(\lambda, 0) = I - \lambda K. \tag{14}$$

As a bifurcation criterion for analytic maps it occurred in [M] and for an affine path of compact vector fields in [T]. It also occurs, in the context of bifurcation criteria via Galerkin approximation, in [F], [We] and [W-W].

The transversality assumption (T) has its origin in the simple-eigenvalue theorem of Crandall and Rabinowitz [C-R] in which $\partial F/\partial x(\lambda, 0)$ again has the form (14), so that (T) means just that the geometric multiplicity of λ_*^{-1} as an eigenvalue of K is odd and coincides with its algebraic multiplicity. The local bifurcation theorem of Westriech [W, Theorem A] follows from the above corollary, with assumption (T). When $\partial F/\partial x(\lambda, 0)$ is affine in λ, assumption (T) has appeared as a local bifurcation condition in a number of different settings (cf. [M, L-M, C-H, W-W, T,F] and the references therein).

In [M], [I1], [S], [Ki], [Ra], and [L-M] various concepts of generalized multiplicity $m(\alpha, \lambda_*) \in \mathbf{N}$ were introduced for paths $\alpha: I \to \Phi_0(X, Y)$ at points $\lambda_* \in I$ which are isolated singular points of α; in each context, additional conditions were imposed on α. In [F-P2], we showed that for each of these multiplicities,

$$\sigma(\alpha, \lambda_*) = (-1)^{m(\alpha, \lambda_*)} \tag{15}$$

and hence each of the corresponding local bifurcation results is a corollary of Theorem 1.

In [E-L] and [E] a generalized multiplicity is introduced for analytic paths of linear Fredholm operators, which is shown, in the context of analytic perturbations, to be an optimal local bifurcation invariant (cf. [I2], also). As observed in [F-P2], (15) also holds for this generalized multiplicity.

Remark 1. We have only considered local bifurcation. By exploiting the additive property of degree, the parity becomes a global bifurcation invariant (cf. [R, F-P1, F-P3, F]).

Remark 2. The assumption in Theorem 1 that F is C^1 can be relaxed. It is clear that the proof carries over if, say, we suppose that

$$F(\lambda, x) = L(\lambda)x + R(\lambda, x),$$

where $L: \mathbf{R} \to \Phi_0(X, Y)$ is continuous, $R(\lambda, 0) = 0$ and $\partial R/\partial x(\lambda, 0) = 0$ for all $\lambda \in \mathbf{R}$ and R is a compact perturbation of a Lipschitz continuous map of small Lipschitz norm, near $[a, b] \times \{0\}$.

Remark 3. The parity assumption in Theorem 1 is stable under suitable changes of variable. For instance, if $\eta: \mathbf{R} \times X \to X$ is C^1, $\eta(\lambda, 0) = 0$ and $\partial\eta/\partial x(\lambda, 0) \in \mathrm{GL}(X, X)$ for $\lambda \in I$, then $\sigma(\partial\widetilde{F}/\partial x(\lambda, 0), [a, b]) = \sigma(\partial F/\partial x(\lambda, 0), [a, b])$, where $\widetilde{F}(\lambda, x) = F(\lambda, \eta(\lambda, x))$.

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