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PERIODIC SOLUTIONS OF SOME LIÉNARD EQUATIONS WITH SINGULARITIES

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ABSTRACT. We consider the forced Liénard equation

$$u'' + f(u)u' + g(t, u) = h(t)$$

together with the boundary conditions

$$u(0) = u(T), \qquad u'(0) = u'(T),$$

where g is continuous on $\mathbf{R} \times (0, +\infty)$ and becomes infinite at $u = 0$. We consider classical solutions as well as generalized solutions that can go into the singularity $u = 0$. The method of approach uses upper and lower solutions and degree theory.

1. INTRODUCTION

Let $h \in L^1(0, T)$, $f \in C(\mathbf{R}, \mathbf{R})$, and $g: \mathbf{R}\times(0, +\infty) \to \mathbf{R}$ be a Carathéodory function, T-periodic in the first variable. We are interested in the existence of solutions to

$$\begin{aligned} &u'' + f(u)u' + g(t, u) = h(t), \\ &u(0) = u(T), \qquad u'(0) = u'(T). \end{aligned} \tag{0}$$

Throughout we assume that:

(A) Given real numbers $0 < a < b$, there exists a function $k \in L^1(0, T)$ such that, if $t \in [0, T]$ and $a \le u \le b$, we have

$$|g(t, u)| \le k(t).$$

Moreover, our assumptions are such that $u = 0$ is a singularity of the "force" g, which becomes unbounded as $u \to 0$.

The corresponding problem in the absence of dissipation has been studied by Lazer and Solimini [4]. We use some of the ideas of that paper to obtain existence results for (0). The paper is organized as follows: In §2 we state and prove a theorem on upper and lower solutions for the $W^{2,1}(0, T)$ setting; such

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a result is a particular case of results obtained by Adje [1] in his doctoral dissertation. §3 deals with attractive forces; Theorem 1 provides classical solutions and Theorem 2 generalized solutions which allow collision with the singularity $u = 0$; the latter are related to a concept introduced by Bahri and Rabinowitz [2]. In §4 we study repulsive forces using degree theory. This approach is very similar to Habets and Sanchez [3] where a vector case is studied using a "strong force" condition. In the scalar case such a "strong force" assumption turns out to be unnecessary.

Throughout the paper we use $\| \ \|_p$ to denote the usual norm in $L^p(0, T)$ $(1 \leq p \leq \infty)$.

2. An auxiliary result involving lower and upper solutions

Let α, β be two continuous functions in $[0, T]$ such that $\alpha(t) \leq \beta(t)$ for all $t \in [0, T]$ and set

$$E = \{(t, x, y) \colon 0 \leq t \leq T, \alpha(t) \leq x \leq \beta(t), y \in \mathbf{R}\}.$$

Let $f \colon E \to \mathbf{R}$ be a Carathéodory function, i.e. a function with the following properties: (i) for each $x \in \mathbf{R}$ and $y \in \mathbf{R}$ the function $f(\cdot, x, y)$, with domain $\{t \in [0, T] \colon (t, x, y) \in E\}$, is measurable; (ii) the functions $f(\cdot, \alpha(\cdot), y)$ and $f(\cdot, \beta(\cdot), y)$ are measurable for each $y \in \mathbf{R}$; (iii) for almost every $t \in [0, T]$, the function $f(t, \cdot, \cdot)$, with domain $\{(x, y) \colon \alpha(t) \leq x \leq \beta(t)$ and $y \in \mathbf{R}\}$ is continuous; (iv) there exists $h \in L^1(0, T)$ such that, for all (t, x) such that $(t, x, 0) \in E$,

$$|f(t, x, 0)| \leq h(t).$$

Proposition 1. *Let E and f be as above and assume in addition that there exists $L \in L^1(0, T)$ such that for any $(t, x, y_1) \in E$ and $(t, x, y_2) \in E$ we have*

$$|f(t, x, y_1) - f(t, x, y_2)| \leq L(t)|y_1 - y_2|. \tag{1}$$

Suppose moreover that $\alpha, \beta \in W^{2,1}(0, T)$ are such that:

$$\text{(a)} \qquad \begin{aligned} &\alpha''(t) \geq f(t, \alpha(t), \alpha'(t)) \quad \text{a.e. in } [0, T], \\ &\alpha(0) = \alpha(T), \qquad \alpha'(0) \geq \alpha'(T); \end{aligned}$$

$$\text{(b)} \qquad \begin{aligned} &\beta''(t) \leq f(t, \beta(t), \beta'(t)) \quad \text{a.e. in } [0, T], \\ &\beta(0) = \beta(T), \qquad \beta'(0) \leq \beta'(T). \end{aligned}$$

Then the periodic boundary value problem

$$\begin{aligned} &x''(t) = f(t, x(t), x'(t)), \\ &x(0) = x(T), \qquad x'(0) = x'(T), \end{aligned} \tag{2}$$

has at least one solution $x(t)$ such that $\alpha(t) \leq x(t) \leq \beta(t)$, for all $t \in [0, T]$.

Remark. According to (a) and (b) we say that $\alpha(t)$ is a lower solution, and $\beta(t)$ is an upper solution, of problem (2). This proposition is a particular case of results obtained by Adje [1]. We give a complete proof for convenience of the reader.

Proof. We adapt the proof given by Mawhin [5, Theorem 1.1]. Using (iv) and (1) we fix a function $m \in L^1(0, T)$ such that for any y with $|y| \leq 1$ we have

$$\max\{|f(t, \alpha(t), y)|, |f(t, \beta(t), y)|, h(t), L(t)\} < m(t). \tag{3}$$

Consider the modified problem

$$\begin{aligned} x'' &= F(t, x, x'), \\ x(0) = x(T), &\qquad x'(0) = x'(T), \end{aligned} \tag{4}$$

where $F\colon [0, T] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is defined by

$$\begin{aligned} F(t, x, y) &= f(t, \alpha(t), y) + m(t)(x - \alpha(t)) \qquad \text{if } x < \alpha(t), \\ &= f(t, x, y) \qquad \text{if } \alpha(t) \leq x \leq \beta(t), \\ &= f(t, \beta(t), y) + m(t)(x - \beta(t)) \qquad \text{if } x > \beta(t). \end{aligned}$$

First we show that any solution of (4) satisfies $\alpha(t) \leq x(t) \leq \beta(t)$ for all $t \in [0, T]$ so that it is in fact a solution of (2). To see this we argue by contradiction. Suppose that $\max(x - \beta) = x(t_0) - \beta(t_0) > 0$. If $t_0 \in (0, T)$ this implies that the functions $x(t) - \beta(t)$ and

$$-|x'(t) - \beta'(t)| + x(t) - \beta(t)$$

are strictly positive in some interval $(t_0 - \varepsilon, t_0 + \varepsilon)$. Thus we obtain a.e. in this interval

$$\begin{aligned} x''(t) &= f(t, \beta(t), x'(t)) + m(t)(x(t) - \beta(t)) \\ &\geq f(t, \beta(t), \beta'(t)) - m(t)|x'(t) - \beta'(t)| + m(t)(x(t) - \beta(t)) \\ &\geq f(t, \beta(t), \beta'(t)) \geq \beta''(t), \end{aligned}$$

where we have used (1) and the fact that $L(t) \leq m(t)$. This contradicts the fact that $x - \beta$ has a local maximum at t_0. If $t_0 = 0$, then since x is T-periodic the maximum is attained at T as well, and we obtain

$$x'(0) - \beta'(0) \leq 0 \leq x'(T) - \beta'(T)$$

which, according to (b), implies $\beta'(0) = \beta'(T)$. Considering the C^1-extensions of $x(t)$ and $\beta(t)$ we reach a contradiction as above. Hence $x(t) \leq \beta(t)$ for all $t \in [0, T]$. Clearly, a similar argument applies to prove that $\alpha(t) \leq x(t)$ for all $t \in [0, T]$.

It remains to prove that (4) has at least one solution. To this end, consider the homotopy

$$\begin{aligned} x'' &= (1 - \lambda)m(t)x + \lambda F(t, x, x'), \\ x(0) = x(T), &\qquad x'(0) = x'(T), \end{aligned} \tag{5}$$

where $0 \leq \lambda \leq 1$. By well-known results, it suffices as in [5] to obtain a priori estimates (in C^1-norm) for the possible solutions $x(t)$ of (5). In fact using (1) and (3) we can write

$$\begin{aligned} |x''(t)| &= (1 - \lambda)m(t)|x(t)| + \lambda[m(t)(1 + |x'(t)|) + m(t)(|x(t)| + M)] \\ &\leq m(t)(M + 1 + |x(t)| + |x'(t)|), \end{aligned}$$

where $M = \max\{\|\alpha\|_\infty, \|\beta\|_\infty\}$. The estimate on $\|x'\|_\infty$ will follow from Gronwall's inequality provided we can bound $\|x\|_\infty$. Let us prove that

$$\|x\|_\infty \leq M + 1. \tag{6}$$

Suppose that this is not the case. Then there exists $t_0 \in [0, T]$ such that

$$\|x\|_\infty = |x(t_0)| > M + 1.$$

If $x(t_0) > 0$, we can fix an interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ where $|x'(t)| < 1$ and $x(t) > \beta(t) + 1$ so that a.e. in this interval we have

$$x''(t) = (1 - \lambda)m(t)x(t) + \lambda[f(t, \beta(t), x'(t)) + m(t)(x(t) - \beta(t))],$$

where according to (3) the bracket is larger than $-m(t) + m(t) = 0$. This shows that, for any $\lambda \in [0, 1]$, we have $x''(t) > 0$ a.e. in $(t_0 - \varepsilon, t_0 + \varepsilon)$. Hence $x(t_0)$ cannot be a maximum. A similar argument applies if $x(t_0) < 0$. This contradiction proves (6). Therefore the proof is complete.

Remark. If the function f is continuous, the estimate on $x'(t)$ may be obtained via a Nagumo condition (cf. [5]). Here we use instead assumption (1) which, in some sense, is less general, but is adequate to our purpose. More general conditions of this type may be found in [1].

3. Attractive forces

"Solutions" of (0) are always understood as (strictly) positive functions such that (0) is satisfied in the usual $W^{2,1}(0, T)$-sense.

Let us introduce the following assumptions:

(G_1) $\lim_{u \to 0^+} g(t, u) = +\infty$ uniformly in $t \in [0, T]$,

(G_2) there exist $R > 0$ and $g_0 \in L^1(0, T)$ such that $g(t, u) \leq g_0(t)$ if $u \geq R$ and $t \in [0, T]$.

Theorem 1. *Assume* (A), (G_1), (G_2) *hold and in addition*

$$\int_0^T g_0(t)\,dt \leq \int_0^T h(t)\,dt, \tag{7}$$

and there exists $M \in \mathbf{R}$ *such that, for all* $t \in [0, T]$,

$$h(t) \leq M. \tag{8}$$

Then the problem (0) *has at least one solution.*

By a (T-periodic) *generalized solution* of (0) we mean a continuous function $u(t)$ such that: (i) $u(0) = u(T)$, (ii) $u(t) \geq 0$ for all $t \in [0, T]$, (iii) the set $u^{-1}(0)$ has zero measure; (iv) given any component J of $[0, T] \backslash u^{-1}(0)$, we have $u \in W^{2,1}(J)$ and $u(t)$ satisfies equation (0) in the $W^{2,1}(J)$-sense. The use of this broader concept of "solution" is useful if we wish to discard the assumption (8).

Theorem 2. *Assume* (A), (G_1), (G_2) *and*

$$\int_0^T g_0(t)\,dt < \int_0^T h(t)\,dt. \tag{9}$$

In addition, let there exist $\varepsilon > 0$ *and* $B \in L^1(0, T)$ *such that, for all* $t \in [0, T]$ *and* $u \in (0, \varepsilon)$, *we have*

$$ug(t, u) \le B(t). \tag{10}$$

Then problem (0) *has a generalized solution.*

Proof of Theorem 1. By condition (G_1) and (8) we can choose a constant $\alpha > 0$ sufficiently small so that for all $t \in [0, T]$

$$g(t, \alpha) \ge M,$$

where M is the upper bound for $h(t)$ in (8). Clearly, $\alpha(t) \equiv \alpha$ is a lower solution of (0). To construct an upper solution, set, for each $b \in L^1(0, T)$,

$$\bar{b} = \frac{1}{T}\int_0^T b(t)\,dt.$$

Fix a constant $C = T\|\varphi\|_1 + R + \alpha$, where $\varphi(t) = (g_0(t) - h(t)) - (\overline{g_0} - \bar{h})$. We claim that there exists a solution $\beta_0 \in W^{2,1}(0, T)$ of the following problem:

$$\begin{gathered} \beta_0'' + f(\beta_0 + C)\beta_0' + \varphi(t) = 0, \\ \beta_0(0) = \beta_0(T), \qquad \beta_0'(0) = \beta_0'(T), \\ \bar{\beta}_0 = 0. \end{gathered} \tag{11}$$

According to well-known results of degree theory, this follows from considering the homotopy

$$\begin{gathered} \beta'' + \lambda f(\beta + C)\beta' + \varphi(t) = 0, \\ \beta(0) = \beta(T), \qquad \beta'(0) = \beta'(T), \\ \bar{\beta} = 0 \end{gathered} \tag{12}$$

$(0 \le \lambda \le 1)$, if we can show that solutions of (12) are bounded, say, in $H^1(0, T)$. Now multiplying (12) by β and integrating we get

$$\|\beta'\|_2^2 = \int_0^T \varphi\beta \le \|\varphi\|_1\|\beta\|_\infty \le \sqrt{T}\|\varphi\|_1\|\beta'\|_2,$$

so that $\|\beta'\|_2 \le \sqrt{T}\|\varphi\|_1$ and finally $\|\beta\|_\infty \le T\|\varphi\|_1$. Thus the claim is proved and moreover we know that (11) has a solution β_0 such that $\|\beta_0\|_\infty \le T\|\varphi\|_1$. Setting $\beta = \beta_0 + C$, it follows that

$$\beta'' + f(\beta)\beta' + \varphi = 0 \tag{13}$$

and β is a T-periodic function such that $\beta(t) \ge R + \alpha$. In particular, using (G_2) and (7) we obtain

$$g(t, \beta(t)) - h(t) \le g_0(t) - h(t) = \varphi(t) + \overline{g_0} - \bar{h} \le \varphi(t)$$

so that, by virtue of (13), β is the desired upper solution. Clearly, equation (0) and the functions α and β satisfy all the requirements in the hypothesis of Proposition 1, so that the proof is complete.

Remark. Condition (G_1) can be weakened in such a way that we are still able to construct a lower solution of (0). In fact replace (G_1) by the following assumption:

$(G_1)'$. There exists $r > 0$ such that for all $u \in (0, r)$ we have $g(t, u) \geq g_0(t)$, and either

$$\overline{g_0} \geq h(t) \quad (t \in [0, T]) \quad \text{and} \quad \|g_0 - \overline{g_0}\|_1 < r/2T,$$

or

$$\overline{g_0} \geq \bar{h} \quad \text{and} \quad \|g_0 - h - (\overline{g_0} - \bar{h})\|_1 < r/2T.$$

It is easy to show (by an argument similar to one used in the above proof) that there exists a solution $\alpha(t)$ of

$$\alpha'' + f(\alpha)\alpha' + g_0 - \overline{g_0} = 0,$$
$$\bar{\alpha} = \frac{r}{2}, \quad \alpha(0) = \alpha(T), \quad \alpha'(0) = \alpha'(T),$$

satisfying in addition $\|\alpha'\|_2 \leq T^{1/2}\|g_0 - \overline{g_0}\|_1$ and $\|\alpha - r/2\|_\infty \leq T\|g_0 - \overline{g_0}\|_1$. It follows from $(G_1)'$ that α takes values in $(0, r)$ and hence it is a lower solution of (0).

Proof of Theorem 2. For each $n \in \mathbf{N}$, let $h_n = \inf(h, n)$. From Theorem 1 it follows that the truncated problem

$$\begin{aligned} &u_n'' + f(u_n)u_n' + g(t, u_n) = h_n(t), \\ &u_n(0) = u_n(T), \qquad u_n'(0) = u_n'(T), \end{aligned} \tag{14}$$

has a solution $u_n \in W^{2,1}(0, T)$ which is strictly positive. To obtain estimates on u_n we first remark that, for n sufficiently large, we cannot have $u_n(t) \geq R$ for all $t \in [0, T]$, since

$$\int_0^T g_0(t) \geq \int_0^T g(t, u_n(t))\,dt = \int_0^T h_n(t)\,dt$$

and this would contradict (9). Hence there exists t_n such that $u_n(t_n) \leq R$. On the other hand

$$\|u_n'\|_2^2 = \int_0^T [g(t, u_n)u_n - h(t)u_n]\,dt$$

which, on the basis of (A), (G_2) and (10) implies

$$\|u_n'\|_2^2 \leq c_1\|u_n\|_\infty + c_2 \leq c_1(R + \sqrt{T}\|u_n'\|_2) + c_2,$$

where c_1, c_2 are positive constants independent of n. Thus for some subsequence, still labelled (u_n), we have

$$u_n \to u \qquad \text{weakly in } H^1(0, T) \text{ and uniformly in } [0, T].$$

Of course, $u(t) \geq 0$ for all $t \in [0, T]$. Set

$$\begin{aligned} \tilde{g}(t) &= g(t, u(t)) \quad \text{if } u(t) > 0, \\ &= +\infty \quad \text{if } u(t) = 0. \end{aligned}$$

Our hypothesis (A) on g and (G_1) show that the sequence of functions $g(t, u_n(t))$ is bounded below by some integrable function. Then using Fatou's lemma we have

$$\int_0^T \tilde{g}(t)\,dt \leq \lim \int_0^T g(t, u_n(t))\,dt = \lim \int_0^T h_n(t)\,dt = \int_0^T h(t)\,dt,$$

showing that $\tilde{g}(t)$ is finite almost everywhere in $(0, T)$. This proves that the set $u^{-1}(0)$ has zero measure. Now let J be any component of $[0, T]\backslash u^{-1}(0)$. Taking an arbitrary test function $\varphi \in C_0^\infty(J)$, we obtain from (14)

$$\int_0^T [-u_n'\varphi' + f(u_n)u_n'\varphi + g(t, u_n)\varphi]\,dt = \int_0^T h_n\varphi$$

and, since $g(t, u_n(t)) \to g(t, u(t))$ in $L^1(\operatorname{supp}\varphi)$,

$$\int_0^T [-u'\varphi' + f(u)u'\varphi + g(t, u(t))\varphi]\,dt = \int_0^T h\varphi.$$

But we know that $g(t, u(t)) = \tilde{g}(t)$ is $L^1(0, T)$ so that $u \in W^{2,1}(J)$ and the result follows.

4. Repelling forces

In this section we consider the problem

$$\begin{aligned} &u'' + f(u)u' + g(u) = h(t), \\ &u(0) = u(T), \qquad u'(0) = u'(T), \end{aligned} \tag{15}$$

where $f \in C(\mathbf{R}, \mathbf{R})$ and $g \in C((0, +\infty), \mathbf{R})$. We introduce the following assumptions:

(F_1) There exists $a > 0$ such that, for every $u > 0$,

$$f(u) \geq a \qquad [\text{respectively } f(u) \leq -a].$$

(G_3) There exists $R > 1$ such that, for every $u > R$,

$$g(u) > \bar{h}.$$

(G_4) There exists $r < 1$ such that, for every $u \in (0, r)$,

$$g(u) < \bar{h}.$$

(G_5)

$$\int_0^1 g(u)\,du = -\infty.$$

Theorem 3. *Assume $h \in L^2(0, T)$ and let f and g be real continuous functions in* $\mathbf{R}$ *and in* $(0, +\infty)$, *respectively, satisfying* (F_1), (G_3), (G_4) *and* (G_5). *Then the problem* (15) *has at least one solution.*

Proof. Consider the homotopy

$$\begin{aligned} &u'' + f(u)u' + g_\lambda(u) = h_\lambda(u), \\ &u(0) = u(T), \qquad u'(0) = u'(T), \end{aligned} \tag{16}$$

where $g_\lambda(u) = (1-\lambda)(\bar{h} + 1 - u^{-1}) + \lambda g(u)$, $h_\lambda(t) = (1-\lambda)\bar{h} + \lambda h(t)$. Note that g_λ satisfies (G_3), (G_4) and (G_5) uniformly in $\lambda \in [0, 1]$ and that $\bar{h}_\lambda = \bar{h}$.

First we obtain an $L^2(0, T)$-bound for the derivative of any solution $u(t)$ of (16). Indeed, multiplication by u' and integration yields

$$a\|\dot{u}\|_2^2 \le \pm \int_0^T f(u)u'^2 = \pm \int_0^T h_\lambda u' \le \|h\|_2 \|u'\|_2 .$$

Next, it follows easily from (G_3) that there exists a t_0 such that

$$u(t_0) < R;$$

this, together with what has just been shown gives an upper bound for all solutions $u(t)$ of (16):

$$u(t) \le u(t_0) + \int_{t_0}^t |u'| < R + T^{1/2}\|h\|_2 a^{-1} \equiv M .$$

Now we are going to obtain a uniform bound on $u'(t)$. To this purpose, extend u as a T-periodic function and let $u(t_1)$ be any extremum of u. For $t \ge t_1$ it turns out that

$$u'(t) = \int_{t_1}^t u'' = -\int_{t_1}^t f(u)u' - \int_{t_1}^t g_\lambda(u) + \int_{t_1}^t h_\lambda .$$

Since $\|u\|_\infty < M$, (G_4) implies that $g_\lambda(u)$ is bounded above. Hence the right-hand side is bounded from below, say

$$u'(t) > -M_1 .$$

Similarly, for $t \le t_1$,

$$u'(t) = \int_t^{t_1} f(u)u' + \int_t^{t_1} g_\lambda(u) + \int_t^{t_1} h_\lambda \le M_1 .$$

This shows that

$$\|u'\|_\infty < M_1 . \tag{17}$$

On the basis of (G_4) we can choose t_2 such that $u(t_2) > r$. Multiplying (16) by u' and integrating in an interval $[t_2, t]$ we get

$$\frac{u'(t)^2}{2} - \frac{u'(t_2)^2}{2} + \int_{t_2}^t f(u)u'^2 + \int_{t_2}^t g_\lambda(u)u' = \int_{t_2}^t h_\lambda u'$$

so that, using the bounds on $u(t)$, $\|u'\|_\infty$ and (17), we see that

$$\int_{t_2}^{t} g_\lambda(u)u' = \lambda \int_{u(t_2)}^{u(t)} g(u)\,du + (1-\lambda)\left[\ln\frac{u(t_2)}{u(t)} + (\bar{h}+1)(u(t)-u(t_2))\right]$$

is bounded. Assumption (G_5) then shows that there exists $\varepsilon > 0$ such that, for any such solution $u(t)$,

$$u(t) > \varepsilon, \qquad t \in [0, T].$$

The estimates thus obtained enable us to apply coincidence degree to the homotopy (16). Define the operators

$$L\colon D(L) \subset H^2(0, T) \to L^2(0, T),$$
$$N\colon [0, 1] \times \overline{\Omega} \to L^2(0, T),$$

where

$$D(L) = \{u \in H^2(0, T)\colon u(0) = u(T),\ u'(0) = u'(T)\},$$
$$Lu = u'' \qquad \text{if } u \in D(L),$$
$$\Omega = \{u \in C^1([0, T])\colon \varepsilon < u(t) < M,\ \|u'\|_\infty < M_1\},$$
$$N(\lambda, u) = h_\lambda - f(u)u' - g_\lambda(u).$$

From well-known properties of the coincidence degree [5] one has

$$d_L(L - N(\lambda, \cdot), \Omega) = d_L(L - N(0, \cdot), \Omega) = \pm d_B(\bar{h} - g_0(u), D) = \pm 1.$$

Here d_B denotes the Brower degree, $D =]\varepsilon, M[$, and we have used the fact that, by virtue of (F_1), the only solution of (16) for $\lambda = 0$ is the constant solution $u = 1$. This shows that (16) is solvable for $\lambda = 1$ and so the proof is complete.

It is easy to see that all the estimates in the above proof remains valid if we replace (F_1) by

(G_6) There exists a constant $C > 0$ such that, for every $t \in [0, T]$ and every $u > 0$,

$$g(u) - h(t) \le C.$$

The only difference is that the bound on $\|u'\|_2$ now follows from the inequalities

$$\int_0^T u'^2 \le \int_0^T (g(u) - h)u \le k\|u\|_1$$

and

$$u(t) \le R + T^{1/2}\|u'\|_2$$

(obtained as in the above proof, using (G_3)). Hence we can state:

Theorem 4. *Assume $h \in L^2(0, T)$ and let f, g be real continuous functions in* $\mathbf{R}$ *and in* $(0, +\infty)$, *respectively, satisfying* (G_3), (G_4), (G_5) *and* (G_6). *Then the problem* (15) *has at least one solution.*

References

1. A. Adje, *Sur et sous solutions dans les équations différentielles discontinues avec conditions aux limites non lineaires*, These de doctorat, Louvain-la-Neuve, 1987.
2. A. Bahri and P. Rabinowitz, *A minimax method for a class of Hamiltonian systems with singular potentials*, CMS Technical Summary Report, 88-8, 1987.
3. P. Habets and L. Sanchez, *Periodic solutions of dissipative dynamical systems with singular potentials*, Differential and Integral Equations (to appear).
4. A. C. Lazer and S. Solimini, *On periodic solutions of nonlinear differential equations with singularities*, Proc. Amer. Math. Soc. **99** (1987), 109–114.
5. J. Mawhin, *Points fixes, points critiques et problèmes aux limites*, Semin. Math. Sup. no. 92, Presses Univ. de Montréal, 1985.

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