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CONVEX FUNCTIONS WITH RESTRICTED CURVATURE

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ABSTRACT. Given $0 \leq R_1 \leq R_2 \leq \infty$, we consider a class of normalized convex functions f in the unit disk $\mathbf{D}$, for which $\partial f(\mathbf{D})$ satisfies a Blaschke Rolling Theorem condition with radii R_1 and R_2. This class contains the convex functions of bounded type. We study the geometry of the image region $f(\mathbf{D})$ and various covering and distortion properties.

1. INTRODUCTION

Let $B(a, \delta)$ be the open disk of radius δ centered at a, and let $\mathbf{D} = B(0, 1)$. As usual, $\mathbf{S}$ denotes the class of normalized univalent functions in $\mathbf{D}$, and we denote by CV the subclass of $\mathbf{S}$ consisting of convex functions. Assume $f \in \mathrm{CV}$. For fixed $r \in (0, 1)$ the curve $C_r: w = f(re^{it})$, $0 \leq t \leq 2\pi$, has radius of curvature at $f(z)$ given by

$$\rho(z) = |zf'(z)| / \operatorname{Re}\{1 + zf''(z)/f'(z)\}.$$

When clarity requires reference to the function f we will write $\rho(f; z)$, and we adopt a similar convention when it is needed in notations which arise later. Let $\rho_M(r)$ and $\rho_m(r)$ denote the maximum and minimum, respectively, of $\rho(z)$ on $|z| = r$. Setting $n(z) = zf'(z)/|zf'(z)|$ (the unit exterior normal to C_r at $f(z)$), Blaschke's Rolling Theorem (e.g., see [3]) implies that

$$B(f(z) - \rho_m(r)n(z), \rho_m(r)) \subseteq f(B(0, r)) \subseteq B(f(z) - \rho_M(r)n(z), \rho_M(r))$$

for each z with $|z| = r$.

Walsh [8] showed that either $\rho_M(r)/r$ is strictly increasing on $(0, 1)$ or $f(z) = z$. He also observed that $\rho_m(r)$ need not be monotone. Let

$$R_2(f) = \lim_{r \to 1} \rho_M(r)$$

and

$$R_1(f) = \liminf_{r \to 1} \rho_m(r).$$

Since $\rho(z)/|z| \to |f'(0)| = 1$ as $z \to 0$, we see from Walsh's result that $1 \leq \rho_M(r)/r \leq R_2(f)$, with equality if and only if $f(z) = z$. For $0 \leq R_1 \leq R_2 \leq \infty$

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and $R_2 \geq 1$ let $\mathrm{CV}(R_1, R_2) = \{f \in \mathrm{CV}: R_1 \leq R_1(f) \leq R_2(f) \leq R_2\}$. These functions were studied by Goodman [1, 2], who called them convex functions of bounded type. Distortion and coefficient results for this class have been obtained by Wirths [9] and Mejia and Minda [4].

In this paper we consider a closely related class of functions whose definition is motivated by the Blaschke Rolling Theorem. Given $0 \leq R_1 \leq R_2 \leq \infty$ and $R_2 \geq 1$, let $\mathrm{CVG}(R_1, R_2)$ be the class of functions f in $\mathbf{S}$ with the property that for each $\eta \in \partial f(\mathbf{D})$ there are open disks $D_1(\eta)$ and $D_2(\eta)$ of radius R_1 and R_2, respectively, such that $\eta \in \partial D_1(\eta) \cap \partial D_2(\eta)$ and

$$D_1(\eta) \subseteq f(\mathbf{D}) \subseteq D_2(\eta). \tag{1}$$

If $R_1 = 0$ or $R_2 = \infty$ we interpret $D_1(\eta)$ or $D_2(\eta)$ to be the empty set or an open half-plane, respectively.

If f is a bounded function in $\mathrm{CVG}(R_1, R_2)$ and $R_1 > 0$, then $\partial f(\mathbf{D})$ is smooth in the sense that it has a continuously turning tangent. Yet we shall see that $\partial f(\mathbf{D})$ need not be a curve of class C^2 and $\rho(z)$ may not be continuous on $\overline{\mathbf{D}}$. In §2 we show that $\mathrm{CV}(R_1, R_2) \subseteq \mathrm{CVG}(R_1, R_2)$. Next, §3 concerns characterizations of $\mathrm{CVG}(R_1, R_2)$ in terms of the geometry of the regions $f(\mathbf{D})$. In §4 we obtain some partial results on the relationship between the largest disk centered at zero which is covered by $f(\mathbf{D})$ and the $\sup\{|f(z)|: z \in \mathbf{D}\}$ when $f \in \mathrm{CVG}(R_1, R_2)$. Finally, we study the order of growth of $|f'|$ and discuss some open questions.

2. Preliminaries

Each f in CV has a spherically continuous extension to $\overline{\mathbf{D}}$. Moreover, there are at most two points on $\partial\mathbf{D}$ at which f takes the value infinity, and except for this occurrence, $f(\partial\mathbf{D})$ is a simple closed curve in the extended complex plane. Since zf' is starlike, $\arg zf'(z)$ has a radial limit everywhere [6]. Thus, though $f(\partial\mathbf{D})$ may not be smooth, $n(\zeta)$ does exist as a radial limit for all $\zeta \in \partial\mathbf{D}$. Let

$$\Phi(t) = \lim_{r \to 1} \arg n(re^{it}).$$

Since Φ is increasing [6], it has at most a countable number of discontinuities, each of which is a simple jump. Moreover, if Φ has a jump at t of magnitude $\alpha\pi$, $\alpha > 0$, and $f(e^{it})$ is finite, then $\alpha < 1$, and we shall say that $\partial f(\mathbf{D})$ has a corner at $f(e^{it})$ with interior angle $(1-\alpha)\pi$.

Lemma 1. *If $f \in \mathrm{CV}$ and $\partial f(\mathbf{D})$ has a corner at $f(e^{it_0})$, then $\rho(re^{it_0}) \to 0$ as $r \to 1$.*

Proof. Assume $f \in \mathrm{CV}$. Pommerenke [6, Lemma 1] showed that

$$zf'(z) = z \exp\left\{(-1/\pi)\int_0^{2\pi} \log(1 - ze^{-it})\, d\Phi(t)\right\},$$

so

$$\rho(z) = |zf'(z)| / \left[(1/2\pi) \int_0^{2\pi} \operatorname{Re}\{(1 + ze^{-it})/(1 - ze^{-it})\}\, d\Phi(t) \right].$$

Suppose Φ has a jump of magnitude $\alpha\pi$ at t_0. If $\delta \in (0, \pi)$, then

$$(1/2\pi) \int_0^{2\pi} \operatorname{Re}\{(1 + re^{i(t_0 - t)})/(1 - re^{i(t_0 - t)})\}\, d\Phi(t)$$
$$\geq (1/2\pi) \int_{t_0 - \delta}^{t_0 + \delta} \operatorname{Re}\{(1 + re^{i(t_0 - t)})/(1 - re^{i(t_0 - t)})\}\, d\Phi(t),$$

and the quantity on the right decreases to $\alpha(1 + r)/2(1 - r)$ as δ decreases to zero. Thus, for $z = re^{it_0}$,

$$\operatorname{Re}\{1 + zf''(z)/f'(z)\} \geq \alpha(1 + r)/2(1 - r). \tag{2}$$

Moreover, Pommerenke [6, p. 212] showed that if $\eta > 0$ and $z = re^{it_0}$, then

$$\ln|zf'(z)| < \ln r + (1/\pi)[\Phi(t_0 + \eta) - \Phi(t_0 - \eta)] \ln[(1 + r)/(1 - r)] + B, \tag{3}$$

where $B = 2\pi / \sin^2(\eta)$. Given $\varepsilon > 0$, fix $\eta > 0$ such that $\Phi(t_0 + \eta) - \Phi(t_0 - \eta) < (\alpha + \varepsilon)\pi$. It then follows from (2) and (3) that

$$\alpha\rho(z) \leq 2re^B [(1 - r)/(1 + r)]^{1 - \alpha - \varepsilon}.$$

Since $\alpha < 1$, we may choose ε so that $1 - \alpha - \varepsilon > 0$, and the conclusion follows.

Let $f_r(z) = f(rz)/r$, $0 < r < 1$. Since $\rho(f_r; z)$ is continuous on $\overline{\mathbf{D}}$, it is easy to see that $R_1(f_r)$ and $R_2(f_r)$ are the minimum and maximum, respectively, of $\rho(f_r; z)$ on $\partial\mathbf{D}$. Then $R_1(f_r) = \rho_m(r)/r$ and $R_2(f_r) = \rho_M(r)/r$, so $R_1(f)$ and $R_2(f)$ are the limit inferior and limit, respectively, of $R_1(f_r)$ and $R_2(f_r)$, as $r \to 1$.

Theorem 1. $\mathrm{CV}(R_1, R_2) \subseteq \mathrm{CVG}(R_1, R_2)$.

Proof. Assume $f \in \mathrm{CV}(R_1, R_2)$, and let $\eta = f(\zeta) \in \partial f(\mathbf{D})$. Applying Blaschke's Rolling Theorem to f_r, we have

$$B(f_r(\zeta) - R_1(f_r)n(r\zeta), R_1(f_r)) \subseteq f_r(\mathbf{D}) \subseteq B(f_r(\zeta) - R_2(f_r)n(r\zeta), R_2(f_r)) \tag{4}$$

for each $\zeta \in \partial\mathbf{D}$. If $w \in f(\mathbf{D})$, then $w \in f_r(\mathbf{D})$ for sufficiently large r. Letting r tend to 1, the right side of (4) yields

$$w \in \overline{B(f(\zeta) - R_2(f)n(\zeta), R_2(f))} \subseteq \overline{B(f(\zeta) - R_2 n(\zeta), R_2)}.$$

Setting $D_2(\eta) = B(f(\zeta) - R_2 n(\zeta), R_2)$, we conclude (since $f(\mathbf{D})$ is open) that $f(\mathbf{D}) \subseteq D_2(\eta)$. This establishes the right side of (1). If $R_1 = 0$, then the left side of (1) is clear, so assume $R_1 > 0$. Let $w \in B(f(\zeta) - R_1(f)n(\zeta), R_1(f))$ and choose a sequence of values of r converging to 1 such that $R_1(f_r) \to R_1(f)$. From (4) we have

$$w \in B(f_r(\zeta) - R_1(f_r)n(r\zeta), R_1(f_r)) \subseteq f_r(\mathbf{D})$$

for all these values of r which are sufficiently near 1, and consequently $w \in f(\mathbf{D})$. Thus,

$$B(f(\zeta) - R_1 n(\zeta), R_1) \subseteq B(f(\zeta) - R_1(f)n(\zeta), R_1(f)) \subseteq f(\mathbf{D}),$$

and $D_1(\eta) = B(f(\zeta) - R_1 n(\zeta), R_1)$ satisfies the left side of (1).

Suppose Ω is a bounded convex region. Given $z \in \Omega$, let ϕ_z denote the conformal mapping of $\mathbf{D}$ onto Ω satisfying $\phi_z(0) = z$ and $\phi_z'(0) > 0$. The hyperbolic metric for Ω has density $\lambda_\Omega(z) = 1/\phi_z'(0)$. The following lemma will be needed in subsequent sections and follows directly from results on λ_Ω in [5, p. 474].

Lemma 2. *Suppose Ω is a bounded convex region that is symmetric about two distinct lines which intersect at z_0. If $z_1 \in \Omega \backslash \{z_0\}$ and $z_t = (1-t)z_0 + tz_1$, $0 < t < 1$, then $\phi_{z_1}'(0) < \phi_{z_t}'(0) < \phi_{z_0}'(0)$.*

3. Geometry of $f(\mathbf{D})$

Assume $0 < R < \infty$, $a, b \in \mathbf{C}$ (the complex numbers), and $|a-b| \le 2R$. If $|a-b| < 2R$, then there are two open disks Δ_1 and Δ_2 of radius R such that $a, b \in \partial\Delta_1 \cap \partial\Delta_2$, and we set $E(a, b; R) = \Delta_1 \cap \Delta_2$. If $|a-b| = 2R$, set $E(a, b; R) = B((a+b)/2, R)$. We shall say that a set $A \subseteq \mathbf{C}$ is R-convex if $\operatorname{diam}(A) \le 2R$ and $E(a, b; R) \subseteq A$ for all choices of a and b in A. Assuming $\operatorname{diam}(A) \le 2R$, we define the R-hull of A to be the smallest R-convex set containing A and denote it by $\operatorname{co}_R(A)$. In the limiting case when $R = \infty$ we set $E(a, b; \infty) = (a, b)$, which yields the usual notions of convexity and convex hull. Mejia and Minda [4] obtained the following result.

Theorem A. *If $1 \le R \le \infty$ and $f \in \mathbf{S}$, then $f \in \mathrm{CV}(0, R)$ if and only if* $\operatorname{diam} f(\mathbf{D}) \le 2R$ *and $E(a, b; R) \subseteq f(\mathbf{D})$ for all $a, b \in f(\mathbf{D})$.*

In the theorem which follows we present characterizations of $\mathrm{CVG}(R_1, R_2)$ in terms of the geometry of $f(\mathbf{D})$. Suppose $0 < R_1 \le R_2 \le \infty$ and let $a, b \in \mathbf{C}$. If $|a-b| \le 2(R_2 - R_1)$, then $\operatorname{diam}(B(a, R_1) \cup B(b, R_1)) \le 2R_2$, and we set $E(a, b; R_1, R_2) = \operatorname{co}_{R_2}(B(a, R_1) \cup B(b, R_1))$. If $R_1 = 0$ and $R_2 > 0$, we set $E(a, b; 0, R_2) = E(a, b; R_2)$.

Theorem 2. *If $0 \le R_1 \le R_2 \le \infty$, $R_2 \ge 1$, and $f \in \mathbf{S}$, then the following are equivalent*:

(i) $f \in \mathrm{CVG}(R_1, R_2)$;

(ii) *$f(\mathbf{D})$ is the intersection of open disks of radius R_2 and, in the case $R_1 > 0$, the union of open disks of radius R_1*;

(iii) $\forall u, v \in f(\mathbf{D})\ \exists a, b \in f(\mathbf{D})$ *such that* $u, v \in E(a, b; R_1, R_2) \subseteq f(\mathbf{D})$;

(iv) *$f \in \mathrm{CV}$, and for each $\zeta \in \partial\mathbf{D}$ for which $f(\zeta)$ is finite,*

$$f(\mathbf{D}) \subseteq B(f(\zeta) - R_2 n(\zeta), R_2),$$

and in the case $R_1 > 0$, $B(f(\zeta) - R_1 n(\zeta), R_1) \subseteq f(\mathbf{D})$.

Remark. If $R_2 = \infty$, then an open disk of radius R_2 is to be interpreted as an open half-plane, and $B(f(\zeta) - R_2 n(\zeta), R_2)$ is to be interpreted as the limiting half-plane as $R_2 \to \infty$.

Proof. (i) $\Rightarrow$ (ii). Assuming $f \in \mathrm{CVG}(R_1, R_2)$, we assert that $f(\mathbf{D}) = \bigcap\{D_2(\eta): \eta \in \partial f(\mathbf{D})\}$. Certainly $f(\mathbf{D}) \subseteq \bigcap\{D_2(\eta): \eta \in \partial f(\mathbf{D})\}$. To obtain the opposite containment, suppose $w \notin f(\mathbf{D})$. Fix $w_0 \in f(\mathbf{D})$ and choose $\eta \in [w_0, w] \bigcap \partial f(\mathbf{D})$. Then $f(\mathbf{D}) \subseteq D_2(\eta)$ implies $w \notin D_2(\eta)$, and thus $w \notin \bigcap\{D_2(\eta): \eta \in \partial f(\mathbf{D})\}$. Now assume $R_1 > 0$ and let $w \in f(\mathbf{D})$. If $\mathrm{dist}(w, \partial f(\mathbf{D})) \geq R_1$, let $D(w) = B(w, R_1)$. If $\mathrm{dist}(w, \partial f\mathbf{D})) < R_1$, choose $\eta \in \partial(f(\mathbf{D}))$ such that $|\eta - w| = \mathrm{dist}(w, \partial f(\mathbf{D}))$ and let $D(w) = D_1(\eta)$. Then $f(\mathbf{D}) = \bigcup\{D(w): w \in f(\mathbf{D})\}$, and (ii) is established.

(ii) $\Rightarrow$ (iii). Let $u, v \in f(\mathbf{D})$. Suppose first that $R_1 = 0$ and let $\{D_\alpha: \alpha \in A\}$ be a collection of open disks of radius R_2 such that $f(\mathbf{D}) = \bigcap\{D_\alpha: \alpha \in A\}$. Since $f(\mathbf{D})$ is convex (being the intersection of open disks), $[u, v] \subseteq f(\mathbf{D})$. Choose a and b on the line determined by u and v such that $[u, v] \subseteq (a, b) \subseteq f(\mathbf{D})$. For each $\alpha \in A$, $(a, b) \subseteq D_\alpha$, and hence $E(a, b; R_2) \subseteq D_\alpha$. Thus, $u, v \in E(a, b; R_2) \subseteq f(\mathbf{D})$. Now assume that $R_1 > 0$, in which case we assume in addition that $f(\mathbf{D})$ is the union of open disks of radius R_1. Choose open disks D_u and D_v of radius R_1 with centers a and b, respectively, such that $u \in D_u \subseteq f(\mathbf{D})$ and $v \in D_v \subseteq f(\mathbf{D})$. Since $f(\mathbf{D})$ lies in a disk of radius R_2, $\mathrm{diam}(D_u \cup D_v) \leq 2R_2$. We claim that $E(a, b; R_1, R_2) = \mathrm{co}_{R_2}(D_u \cup D_v) \subseteq f(\mathbf{D})$. Indeed, since $\mathrm{CVG}(R_1, R_2) \subseteq \mathrm{CVG}(0, R_2)$, we may apply the earlier part of the proof to conclude that $E(a', b'; R_2) \subseteq f(\mathbf{D})$ for all $a' \in D_u$ and $b' \in D_v$, and the conclusion follows.

(iii) $\Rightarrow$ (iv). Since $E(a, b; R_1, R_2)$ is a convex set, (iii) implies that $f(\mathbf{D})$ is convex. It is implicit in (iii) that $\mathrm{diam}\, f(\mathbf{D}) \leq 2R_2$. Fix $\zeta \in \partial\mathbf{D}$ such that $f(\zeta) \neq \infty$, and consider $w \in f(\mathbf{D})$. Suppose first that $R_1 > 0$. For sufficiently small $\varepsilon > 0$, $f(\zeta) - \varepsilon n(\zeta) \in f(\mathbf{D})$, so there are points a_ε and b_ε such that $w, f(\zeta) - \varepsilon n(\zeta) \in E(a_\varepsilon, b_\varepsilon; R_1, R_2) \subseteq f(\mathbf{D})$. Let ε tend to zero though a sequence of values for which a_ε and b_ε are convergent, say to a and b, respectively. Then $w \in \overline{E(a, b; R_1, R_2)}$, and $f(\zeta) \in \partial E(a, b; R_1, R_2)$. Since $R_1 > 0$, $\partial f(\mathbf{D})$ has a tangent at $f(\zeta)$, and it follows from the shape of $E(a, b; R_1, R_2)$ that

$$B(f(\zeta) - R_1 n(\zeta), R_1) \subseteq E(a, b; R_1, R_2) \subseteq B(f(\zeta) - R_2 n(\zeta), R_2).$$

Thus, $B(f(\zeta) - R_1 n(\zeta), R_1) \subseteq f(\mathbf{D})$, and $w \in \overline{B(f(\zeta) - R_2 n(\zeta), R_2)}$ for all $w \in f(\mathbf{D})$. Since $f(\mathbf{D})$ is open we conclude that $f(\mathbf{D}) \subseteq B(f(\zeta) - R_2 n(\zeta), R_2)$. Now consider the case $R_1 = 0$. The preceding argument yields $f(\mathbf{D}) \subseteq B(f(\zeta) - R_2 n(\zeta), R_2)$ when $\partial f(\mathbf{D})$ has a tangent at $f(\zeta)$, and this occurs for all but a countable number of points $\zeta \in \partial\mathbf{D}$. Suppose $\partial f(\mathbf{D})$ has a corner at $f(\zeta)$, $\zeta = e^{i\theta}$, and let $n(\zeta^+)$ and $n(\zeta^-)$ denote the limits of $n(e^{it})$ as t tends to θ from the right and left, respectively. From the previous case we

conclude that

$$f(\mathbf{D}) \subseteq B(f(\zeta) - R_2 n(\zeta^+), R_2) \cap B(f(\zeta) - R_2 n(\zeta^-), R_2). \tag{5}$$

Since, $\arg n(\zeta^-) \leq \arg n(\zeta) \leq \arg n(\zeta^+)$, the right side of Equation (5) lies in $B(f(\zeta) - R_2 n(\zeta), R_2)$.

Finally, it is clear that (iv) implies (i).

The following result occurs in the proof of (iii) implies (iv) in Theorem 2, and we record it here for emphasis.

Corollary. *Suppose* $f \in \mathrm{CVG}(0, R_2)$ *and* $\partial f(\mathbf{D})$ *has a corner at* $f(\zeta)$, $\zeta = e^{it_0}$. *Then*

$$f(\mathbf{D}) \subseteq B(f(\zeta) - R_2 n(\zeta^+), R_2) \cap B(f(\zeta) - R_2 n(\zeta^-), R_2),$$

where $n(\zeta^+)$ *and* $n(\zeta^-)$ *denote the limits of* $n(e^{it})$ *as* $t \to t_0$ *from the right and left, respectively.*

4. Covering properties

Given f in CV, let $d(f)$ and $M(f)$ be the minimum and maximum, respectively, of $|f(z)|$ on $|z| = 1$. In general, $1/2 \leq d(f) \leq M(f) \leq \infty$. Goodman [1] considered the problems of finding $\inf d(f)$ and $\sup M(f)$ for f in $\mathrm{CV}(0, R)$, and he found the following function and its rotations to be extremal for both problems.

Example. If $1 \leq R < \infty$, let $k_R(z) = z/(1 - z\sqrt{1 - 1/R})$, and if $R = \infty$, let $k_\infty(z) = z/(1 - z)$. Then $k_R(\mathbf{D})$ is $B(\sqrt{R^2 - R}, R)$ or $\{z\colon \operatorname{Re} z > -1/2\}$, depending on whether $R < \infty$ or $R = \infty$, respectively. To avoid the need to distinguish between the two cases, $R < \infty$ and $R = \infty$, we shall agree, in the case $R = \infty$, to interpret $B(\sqrt{R^2 - R}, R)$ to be $\{z\colon \operatorname{Re} z > 1/2\}$ and quantities such as $R - \sqrt{R^2 - R}$ to be their limits (in this case $1/2$) as $R \to \infty$. Then for $1 \leq R \leq \infty$ we have $R_1(k_R) = R = R_2(k_R)$, $d(k_R) = R - \sqrt{R^2 - R}$, and $M(k_R) = R + \sqrt{R^2 - R}$.

The conclusions of the following theorem were obtained by Goodman [1] for the class $\mathrm{CV}(R_1, R_2)$. The proof for $\mathrm{CVG}(R_1, R_2)$ is virtually the same and will not be repeated.

Theorem B. *If* $f \in \mathrm{CVG}(R_1, R_2)$, *then*

$$d(f) \geq R_2 - \sqrt{R_2^2 - R_2} \tag{6}$$

and

$$M(f) \leq R_2 + \sqrt{R_2^2 - R_2}. \tag{7}$$

Equality occurs in (6) *or* (7) *if and only if* f *is a rotation of* k_{R_2}. *Moreover, if* $1 \le R_1 \le \infty$, *then*

$$d(f) \le R_1 - \sqrt{R_1^2 - R_1}, \tag{8}$$

with equality if and only if f *is a rotation of* k_{R_1}.

We now look at the effect on $M(f)$ when $f \in \mathrm{CVG}(R_1, R_2)$ and f covers the disk $B(0, d)$, where $R_2 - \sqrt{R_2^2 - R_2} < d < 1$. We shall need to also assume that $R_1 \le d$. The case $R_2 = \infty$ is somewhat special in that functions in $\mathrm{CVG}(R_1, \infty)$ need not be bounded, and we begin by discussing some of the features of this situation.

Given $r > 0$, let g_r be the conformal mapping of $\mathbf{D}$ onto $\mathrm{co}(B(0, r) \cup (0, \infty))$ such that $g_r(0) = 0$ and $g_r'(0) > 0$. If $0 < r_1 < r_2$, then g_{r_1} is properly subordinate to g_{r_2}, and consequently $g_r'(0)$ is a strictly increasing function of r. If $r \le \pi/4$, then $g_r \prec (1/2)\log[(1+z)/(1-z)]$ and $g_r'(0) < 1$, whereas $r \ge 1$ implies $z \prec g_r$ and $g_r'(0) > 1$. Thus, there is a unique $r^* \in (\pi/4, 1)$ such that $g^* = g_{r^*} \in \mathrm{CV}$.

Lemma 3. *Assume* $f \in \mathrm{CV}$. *If* $d(f) > r^*$, *then* f *is bounded. If* $d(f) = r^*$ *and* f *is not bounded, then* f *is a rotation of* g^*.

Proof. Suppose $d(f) \ge r^*$ and f is not bounded. Choose $\{w_n\}_1^\infty \subseteq f(\mathbf{D})$ such that $w_n \to \infty$ and $\arg w_n \to \beta$. By a rotation we need only consider $\beta = 0$, in which case $\mathrm{co}(B(0, d(f)) \cup (0, \infty)) \subseteq f(\mathbf{D})$. Thus, $g^* \prec f$, and, since both functions are normalized, we conclude that $g^* = f$ and $d(f) = r^*$.

Now, suppose $0 \le R_1 \le R_2 < \infty$, $R_2 > 1$, $R_2 - \sqrt{R_2^2 - R_2} < d < 1$, and $R_1 \le d$. Let $a \in [R_1 - d, 2R_2 - R_1 - d]$ and let U_a be $B(0, d) \cup B(a, R_1)$ or $B(0, d) \cup (0, a)$ depending on whether $R_1 > 0$ or $R_1 = 0$. Then $\operatorname{diam} U_a \le 2R_2$, and there is a unique conformal map φ_a of $\mathbf{D}$ onto $\mathrm{co}_{R_2}(U_a)$ such that $\varphi_a(0) = 0$ and $\varphi_a'(0) > 0$. If $a \in [R_1 - d, d - R_1]$, then $\varphi_a(\mathbf{D}) = B(0, d)$ and $\varphi_a'(0) = d < 1$. Since $R_1 \le d$, the regions $\varphi_a(\mathbf{D})$ expand as a increases, and consequently $\varphi_a'(0)$ is a strictly increasing function of a on $[d - R_1, 2R_2 - R_1 - d]$. If $a = 2R_2 - R_1 - d$, then $\varphi_a(\mathbf{D}) = B(R_2 - d, R_2)$, and since $d > R_2 - \sqrt{R_2^2 - R_2}$, Lemma 2 implies that $\varphi_a'(0) > k_{R_2}'(0) = 1$. Thus, there is a unique $a \in [d - R_1, 2R_2 - R_1 - d]$ for which $\varphi_a'(0) = 1$. This normalized function, denoted hereafter by h_d, is in $\mathrm{CVG}(R_1, R_2)$.

Now, consider $R_2 = \infty$. In this case we choose $a \in [R_1 - d, \infty)$ and define φ_a as before. As long as $d > r^*$, $\lim_{a\to\infty} \varphi_a'(0) = g_d'(0) > 1$, and again there will be a unique value of a for which $\varphi_a'(0) = 1$. This normalized function will also be denoted by h_d.

Theorem 3. *Assume* $0 \le R_1 \le d$, $R_2 > 1$, *and* $R_2 - \sqrt{R_2^2 - R_2} < d < 1$. *Let* $f \in \mathrm{CVG}(R_1, R_2)$ *and suppose* $d(f) \ge d$. *If* $R_2 < \infty$, *then* $M(f) \le M(h_d)$,

with equality if and only if f is a rotation of h_d. If $R_2 = \infty$, then the same conclusions hold as long as $d > r^*$.

Proof. In either case, choose $\zeta \in \partial\mathbf{D}$ such that $|f(\zeta)| = M(f)$. By a rotation it suffices to consider $f(\zeta) > 0$, in which case $n(\zeta) = 1$. Then,

$$B(f(\zeta) - R_1, R_1) \subseteq f(\mathbf{D}),$$

and consequently $\mathrm{co}_{R_2}(B(0, d) \cup B(f(\zeta) - R_1, R_1)) \subseteq f(\mathbf{D})$. If $f(\zeta) \geq M(h_d)$, then h_d is subordinate to f. But, $f'(0) = 1 = h_d'(0)$, so $f = h_d$.

If $R_2 = \infty$, $1/2 \leq d \leq r^*$, and $f \in \mathrm{CVG}(R_1, R_2)$, then $f(\mathbf{D})$ may not be bounded, but there is a restriction on how large a sector f may cover. For each $\theta \in (0, \pi]$, let σ_θ be the conformal map of $\mathbf{D}$ onto $\mathrm{co}(B(0, d) \cup \{z : |\arg z| < \theta/2\})$ such that $\sigma_\theta(0) = 0$ and $\sigma_\theta'(0) > 0$. Let $\sigma_0 = g_d$. Then $\sigma_\theta'(0)$ is a strictly increasing function of θ on $[0, \pi]$. Moreover, $d \geq 1/2$ implies $z/(1-z) \prec \sigma_\pi$ and $\sigma_\pi'(0) \geq 1$, with equality only when $d = 1/2$. Also, $d \leq r^*$ implies $\sigma_0 \prec g^*$ and $\sigma_0'(0) \leq 1$, with equality only when $d = r^*$. Thus, there is a unique $\theta(d)$ in $[0, \pi]$ such that $\sigma_{\theta(d)}'(0) = 1$, and we denote this function by s_d.

Theorem 4. *Suppose $1/2 \leq d < r^*$, $f \in \mathrm{CVG}(R_1, \infty)$, $R_1 \leq d$, and $d(f) \geq d$. If $f(\mathbf{D})$ contains an open sector of angle $\theta(d)$, then f is a rotation of s_d.*

Proof. Suppose $f(\mathbf{D})$ contains an open sector Σ with vertex angle $\theta(d)$. By a rotation we may assume that Σ is symmetric with respect to the real axis and contains $(0, \infty)$, in which case $s_d \prec f$. Thus, $f = s_d$.

5. Growth of $|f'|$

The following result, due to Pommerenke [6, Theorem 1], shows that for convex functions the growth of $M(r, f') = \max\{|f'(z)| : |z| = r\}$ is governed by the magnitude of the largest jump of Φ.

Theorem C. *If $f \in \mathrm{CV}$ and $\alpha\pi = \max\{\Phi(t^+) - \Phi(t^-) : 0 \leq t \leq 2\pi\}$, then*

$$M(r, f') \geq 1/4(1-r)^\alpha \tag{9}$$

and

$$\alpha = \lim_{r \to 1} \log M(r, f')/\log(1/(1-r)). \tag{10}$$

If $f \in \mathrm{CVG}(R_1, R_2)$, $f(\mathbf{D})$ is bounded, and $R_1 > 0$, then Φ is continuous and $\alpha = 0$. Hence we consider the case $R_1 = 0$, $R_2 = R \in [1, \infty)$, and proceed to determine the largest possible jump of Φ. First we discuss the function in $\mathrm{CVG}(0, R)$ which exhibits the extremal behavior.

Example. Suppose $\alpha \in [0, 1)$, let $\phi(z) = (1+z)/(1-z)$, and consider $f_\alpha(z) = \phi^{-1}(\phi(z)^{1-\alpha})/(1-\alpha)$. Elementary calculations show that $f(\mathbf{D}) = E(a, b; R)$, where $a = -(1-\alpha)^{-1}$, $b = (1-\alpha)^{-1}$, and

$$R = 1/(1-\alpha)\sin[(1-\alpha)\pi/2]. \tag{11}$$

Thus, $f \in \mathrm{CVG}(0, R)$. For each $R \in [1, \infty)$ there is a unique $\alpha = \alpha(R) \in [0, 1)$ determined implicitly by (11), and wet set $f_R = f_{\alpha(R)}$. The function $\Phi(f_R, t)$ is continuous on $(0, \pi) \cup (\pi, 2\pi)$ and has a jump discontinuity of magnitude $\alpha(R)\pi$ at 0 and at π.

Theorem 5. *If $1 \leq R < \infty$ and $f \in \mathrm{CVG}(0, R)$, then $\Phi(t^+) - \Phi(t^-) \leq \alpha(R)\pi$ for all t, and equality occurs for some t if and only if f is a rotation of f_R.*

Proof. Fix $\zeta = e^{it}$ and assume $\Phi(t^+) - \Phi(t^-) = \beta\pi > 0$. By the corollary to Theorem 2,

$$f(\mathbf{D}) \subseteq \Omega = B(f(\zeta) - Rn(\zeta^+), R) \cap B(f(\zeta) - Rn(\zeta^-), R).$$

Note that $\Omega = E(f(\zeta), b; R)$ for an appropriate choice of $b \in \mathbf{C}$, and the interior angle of Ω at $f(\zeta)$ is $(1 - \beta)\pi$. There exist $\mu, \nu \in \mathbf{C}$, $|\mu| = 1$, such that $G = (\mu f_R + \nu)(\mathbf{D}) = E(f(\zeta), c; R)$, where c lies on the ray from $f(\zeta)$ through b. Now suppose $\beta > \alpha(R)$. Then $|c - f(\zeta)| > |b - f(\zeta)|$ and $f(\mathbf{D}) \subseteq \Omega \subseteq G$. Let g be the conformal map of $\mathbf{D}$ onto G such that $g(0) = f(0)$ and $g'(0) > 0$. Then $\beta > \alpha(R)$ implies f is properly subordinate to g, and consequently $1 = f'(0) < g'(0)$. But Lemma 2 gives $g'(0) \leq |(\mu f_R + \nu)'(0)| = 1$, a contradiction. Thus, $\beta \leq \alpha(R)$, and the conclusion follows. If $\beta = \alpha(R)$, then the same considerations yield $f(\mathbf{D}) = G$ and $f(0) = 0 = (\mu f_R + \nu)(0)$, so f is a rotation of f_R.

Corollary. *If $1 \leq R < \infty$ and $f \in \mathrm{CVG}(0, R)$, then*

$$M(r, f') = O((1 - r)^{-\alpha(R)}).$$

Proof. Suppose $f \in \mathrm{CVG}(0, R)$ and $f \neq f_R$. By the previous theorem, $\beta = \max\{\Phi(t^+) - \Phi(t^-) : 0 \leq t \leq 2\pi\}/\pi < \alpha(R)$. If ε is chosen so that $\beta + \varepsilon < \alpha(R)$, then (10) gives $M(r, f') = O((1 - r)^{-(\beta+\varepsilon)}) = o((1 - r)^{-\alpha(R)})$. Now, consider f_R. A brief calculation yields

$$f_R'(z) = 4/[(1 + z)^{1-\alpha(R)} + (1 - z)^{1-\alpha(R)}]^2 (1 - z^2)^{\alpha(R)}.$$

The quantity $(1 + z)^{1-\alpha(R)} + (1 - z)^{1-\alpha(R)}$ has a positive minimum modulus m on $\overline{\mathbf{D}}$, so $|f_R'(z)| \leq 4/m^2(1 - |z|^2)^{\alpha(R)}$, and hence $M(r, f_R') = O((1 - r)^{-\alpha(R)})$.

6. Comments

It follows readily from classical results in differential geometry (e.g., see [7]) that if $f \in \mathrm{CVG}(R_1, R_2)$ and f is analytic on $\overline{\mathbf{D}}$, then $f \in \mathrm{CV}(R_1, R_2)$, but it remains to be determined whether or not $\mathrm{CVG}(R_1, R_2) = \mathrm{CV}(R_1, R_2)$. Mejia and Minda [4] have shown that this is the case when $R_1 = 0$, and we expect that it is also true for $R_1 > 0$.

Very little seems to be known about the behavior of $\rho_m(r)$ as $r \to 1$. Although Walsh noted that it need not be monotone, it would be interesting to know if $\rho_m(r)$ has a limit as $r \to 1$, or if, perhaps, it has at most a finite number of local extrema on $(0, 1)$.

The results in Theorem 3 are incomplete due to the assumption $R_1 \leq d$. If $R_1 > d$, then for an appropriate choice of $a \in [R_1 - d, 2R_2 - R_1 - d]$ there is a normalized conformal mapping of $\mathbf{D}$ onto $\mathrm{co}_{R_2}(B(R_1 - d, R_1) \cup B(a, R_1))$, and this may be the extremal function. However, in this case we do not get the subordination used in the proof of Theorem 3 because when f is rotated so that $M(f) = f(\zeta) > 0$, the point on $\partial f(\mathbf{D})$ nearest to $z = 0$ may not be $-d$.

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