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PROJECTIVE STRUCTURES ON REDUCTIVE HOMOGENEOUS SPACES

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ABSTRACT. The aim of this work is to give a more direct and "geometric" proof of a theorem of Agaoka, that on a reductive homogeneous space G/K, every G-invariant projective structure admits a G-invariant affine connection. This connection can be chosen uniquely, subject to being torsionfree and satisfying one extra condition.

INTRODUCTION

Agaoka in [1] proved that every invariant flat projective structure on a reductive homogeneous space admits an invariant affine connection; later, Agaoka communicated to me by letter that using the same technique with some modifications the same result could also be obtained in the not-flat case. The aim of this work is to give a more direct and "geometric" proof of this fact, providing more information about the choice of the invariant connection. The author heartily thanks Prof. Agaoka for his attention and kindness.

1. PRELIMINARIES

We shall give a brief review about projective connections; for more details and a precise exposition we refer to S. Kobayashi and T. Nagano [3] and N. Tanaka [7].

Let M be an n-dimensional C^∞-manifold and $P^2(M)$ its 2-frames bundle with structure group $G^2(n)$; we represent the real projective space as homogeneous space $\mathbf{RP}^n = \mathbf{PGL}(n)/H$ where H is the isotropy subgroup of $\mathbf{PGL}(n)$ at the point $o = (1, 0, \ldots, 0)$.

A *projective structure* $\mathscr{P}$ on M is a principal subbundle P of $P^2(M)$ whose structure group is H (one can show that there is a monomorphism of H into $G^2(n)$). The Lie algebra of $\mathbf{PGL}(n)$ is $\mathbf{sl}(n+1)$ and has a graded Lie algebra structure

$$\mathbf{sl}(n+1) = \mathbf{g}_{-1} \oplus \mathbf{g}_0 \oplus \mathbf{g}_1$$

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with $\mathbf{g}_{-1} \cong \mathbf{R}^n$, $\mathbf{g}_0 \cong \mathbf{gl}(n, \mathbf{R})$, $\mathbf{g}_1 \cong \mathbf{R}^{n*}$ and $\mathbf{g}_0 \oplus \mathbf{g}_1 \cong \mathbf{h}$ where $\mathbf{h}$ is the Lie algebra of H. For later use we note that there are monomorphisms $\phi\colon \mathbf{GL}(n, \mathbf{R}) \to H$ and $\exp\colon \mathbf{R}^{n*} \to H$ given in matrix notation by

$$\phi\colon A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \qquad \exp\colon E \mapsto \begin{pmatrix} 1 & E \\ 0 & Id \end{pmatrix}$$

for every $A \in \mathbf{GL}(n, \mathbf{R})$ and every $E \in \mathbf{R}^{n*}$, so that every element $h \in H$ is expressible in an unique way as $h = \phi(A)\exp(E)$ for some $A \in \mathbf{GL}(n, \mathbf{R})$ and $E \in \mathbf{R}^{n*}$.

Given a projective structure $\mathscr{P}$ on M, one can construct a 1-form ω (called a *Cartan connection*) on the associated bundle P such that

(a) $R_a^*\omega = \mathrm{ad}(a^{-1})\omega \quad \forall a \in H$,

(b) if $\omega(X) = 0$ then $X = 0$.

We will denote by ω_i the $\mathbf{g}_i$-component of ω. If we define the 2-form Ω by means of the structure equation

$$d\omega + \tfrac{1}{2}[\omega, \omega] = \Omega$$

then by conditions (a) and (b) we can express Ω_0 as follows:

$$\Omega_0 = \sum_{l, k=1, \dots, n} K^i_{jlk} \omega^l \wedge \omega^k$$

where $(\omega^i)_{i=1, \dots, n}$ denote the components of ω_{-1}.

It can be proved that there is a unique Cartan connection on P, called the *normal* Cartan connection, such that

(c) (ω_{-1}, ω_0) is the restriction to P of the canonical 1-form θ of $P^2(M)$,

(d) $\sum_{i=1, \dots, n} K^i_{jik} = 0 \quad \forall j, k = 1, \dots, n$.

If we denote by $P^1(M)$ the bundle of linear frames on M, we have the following fundamental theorem (see [3]).

Theorem 1.1. *If P is a projective structure and $j\colon P^1(M) \to P$ is a bundle monomorphism corresponding to the monomorphism ϕ, then $j^*\omega_0$ defines a linear torsionfree connection on $P^1(M)$ and $j^*\omega_{-1}$ is the canonical form on $P^1(M)$. To each linear torsionfree connection Γ on M, one can associate a projective structure P and a bundle monomorphism $j\colon P^1(M) \to P$ such that Γ is the connection defined by $j^*\omega_0$; moreover two linear torsionfree connections define the same system of geodesics up to parametrization iff they define the same projective structure P.*

When two linear torsionfree connections define the same projective structure P, we say that they belong to P. Moreover it can be proved (see [2]) that two linear torsionfree connections ∇_1, ∇_2 belong to the same projective structure iff there exists a 1-form φ such that for every two vector fields X, Y on M

$$\nabla_{1X} Y = \nabla_{2X} Y + \varphi(X)Y + \varphi(Y)X. \tag{1.1}$$

2. PROJECTIVE STRUCTURES ON REDUCTIVE HOMOGENEOUS SPACES

We start establishing the following theorem, whose proof is quite simple:

Theorem 2.1. *Let G be a Lie group and $\mathscr{P}$ a projective structure on G that is G-invariant. Then there exists an affine torsionfree G-invariant connection belonging to the projective structure $\mathscr{P}$.*

Proof. A projective structure $\mathscr{P}$ being given, we have a principal fibre bundle $P(G, H)$ with structure group $H = \mathbf{PGL}(n, \mathbf{R})_o$ $(n = \dim G)$ that is a subbundle of the bundle of 2-frames $P^2(G)$. Since G acts as a group of automorphisms for the projective structure P, the lift of each transformation $L_g (g \in G)$ to $P^2(G)$ is an automorphism of the bundle $P(G, H)$: if we fix a 2-frame $u_o \in P(G, H)$ lying over $1_G \in G$ and consider

$$\Phi_{u_o}: G \to P(G, H),$$
$$g \mapsto L_g^2(u_o)$$

we get an imbedding of G into $P(G, H)$ (see [3]) such that, if π is the projection of $P(G, H)$ onto G,

$$\pi(\Phi_{u_o}(g)) = \pi(L_g^2(u_o)) = L_g(1_G) = g.$$

This means that Φ_{u_o} yields a section over G of the bundle $P(G, H)$, so that the map λ

$$\lambda: P(G, H) \to G \times H,$$
$$u \mapsto (\pi(u), h)$$

(where $h \in H$ is the unique element of H such that $uh = L^2_{\pi(u)}(u_o)$) turns out to be a bundle-isomorphism; moreover the action of G on $G \times H$ induced by the map λ is given simply by left translation of the first coordinate, as one can easily check.

So $\lambda^{-1}(G \times GL(n, \mathbf{R}))$ is a subbundle of $P(G, H)$ that corresponds to a bundle of linear frames $P^1(G)$ over G; this subbundle is obviously G-invariant. If ω denotes the Cartan normal connection form on $P(G, H)$ that is G-invariant, then ω induces a linear torsionfree connection ∇ on $P^1(G)$ that is automatically G-invariant; moreover since $P^1(G)$ is a subbundle of $P(G, H)$, the connection ∇ belongs to the given projective structure and we are done. Q.E.D.

Another proof of the previous result has been given by Nomizu-Pinkall [5].

We now want to generalize the previous theorem in the following direction: let $M = G/K$ be a homogeneous space, where G is a Lie group and K is a closed subgroup of G. We will assume that G/K is reductive, that is, if $\mathbf{g}$ and $\mathbf{k}$ are the Lie algebras of G and K, respectively, there is an $\mathrm{ad}(K)$-invariant subspace $\mathbf{m}$ of $\mathbf{g}$ so that

$$\mathbf{g} = \mathbf{k} \oplus \mathbf{m}. \tag{2.1}$$

With respect to the given decomposition (2.1), we denote by D the canonical G-invariant connection on $M = G/K$ (see [4], vol. II, p. 192); moreover $\widetilde{D}$ will indicate the canonical torsionfree G-invariant affine connection on M given by

$$\widetilde{D}_X Y = D_x Y + \tfrac{1}{2} S(X, Y)$$

where X, Y are vector fields on M, and S is the torsion tensor of D.

Theorem 2.2. *Let $M = G/K$ be a reductive homogeneous space and $\mathscr{P}$ a G-invariant projective structure; then there is a torsionfree affine connection Γ on M that is G-invariant and that belongs to $\mathscr{P}$. Moreover Γ can be chosen in a unique way so that, if ∇ is the covariant derivative corresponding to Γ, then the following condition $(^*)$ is fulfilled*

$$(*) \qquad \operatorname{tr}\{Y \mapsto \nabla_X Y - D_X Y\} = 0$$

for every vector field X on M. The skew symmetric part A of the Ricci tensor of this connection is given by

$$2A(X, Y) = \operatorname{tr} R(X, Y)$$

$(X, Y$ vector fields on $M)$ where R is the curvature tensor of D.

Proof. The projective structure $\mathscr{P}$ corresponds to a subbundle $P = P(M, H)$ of $P^2(M)$ with structure group $H = \mathbf{PGL}(n, \mathbf{R})_o$ $(n = \dim M)$ and projection map π. For every $u \in P$ we get an imbedding of G into P given by:

$$\Phi_u : G \to P,$$
$$g \mapsto g^{(2)}(u),$$

where $g^{(2)}$ is the lift of the transformation g of M to an automorphism of the bundle P. We denote with $p: G \to G/K$ the projection; then for every $u \in P$ we define a subspace Q_u of TP_u as follows:

$$Q_u = \Phi_{u_*}(\operatorname{ad}(g)\mathbf{m}), \tag{2.3}$$

where g is any element of G with $p(g) = \pi(u)$; the definition of Q_u is well-posed since the subspace $\operatorname{ad}(g)\mathbf{m}$ does not depend on the choice of the element g with $p(g) = \pi(u)$ thanks to the reductivity. We observe that

$$\pi_*|_{Q_u} : Q_u \to TM_{\pi(u)}$$

is an isomorphism: indeed if $\pi_*(X) = 0$ with $X \in Q_u$, then $X \in \operatorname{ad}(g)\mathbf{m} \cap \operatorname{ad}(g)\mathbf{k} = (0)$. Moreover we note that the distribution Q is H-invariant since for every $g \in G$ we have that $g^{(2)}$ commutes with the action of H.

Let ω be the normal Cartan connection on P, with values in $\mathbf{sl}(n+1, \mathbf{R}) = \mathbf{g}_{-1} \oplus \mathbf{g}_0 \oplus \mathbf{g}_1$. If we fix a point u_o in P there exist unique tangent vectors $(X_i)_{i=1,\dots,n} \in Q_{u_o}$ such that

$$\omega_{-1}(X_i) = e_i,$$

where $(e_i)_{i=1,\dots,n}$ is the standard basis of $\mathbf{g}_{-1} = \mathbf{R}^n$: that is because $\pi_*|_{Q_{u_o}}$ is an isomorphism onto $TM_{\pi(u_o)}$.

We now construct $\xi \in \mathbf{R}^{n*}$ in the following way:

$$\xi^k = \frac{1}{n+1} \operatorname{tr} \omega_0(X_k)$$

and we claim that if we put $h = \exp(\xi) \in H$, then

$$\operatorname{tr} \omega_0|_{u_o h}(X) = 0 \qquad \forall X \in Q_{u_o h}. \tag{2.2}$$

Indeed we have that $Q_{u_o h} = R_h(Q_{u_o})$ and

$$\omega_0|_{u_o h}(R_h X) = \omega_0|_{u_o}(X) - [\xi, \omega_{-1}|_{u_o}(X)].$$

So (1.2) is true iff

$$\operatorname{tr}(\omega_0|_{u_o}(X_i) - [\xi, \omega_{-1}|_{u_o}(X_i)]) = 0 \qquad \forall i = 1, \dots, n$$

and this follows from the very definition of ξ.

Now let $h' \in H$ such that

$$\operatorname{tr} \omega|_{u_o h'}(X) = 0 \qquad \forall X \in Q_{u_o h'}$$

and note that $u_o h' = (u_o h) h^{-1} h'$; we have that

$$\operatorname{tr} \omega_0|_{u_o h'}(R_{h^{-1}h'*} X) = 0 \qquad \forall X \in Q_{u_o h};$$

hence if $h^{-1}h' = \phi(a)\exp(\eta)$ for some $a \in \mathbf{GL}(n, \mathbf{R})$ and $\eta \in \mathbf{R}^{n*}$,

$$\operatorname{tr}([\operatorname{ad}(\exp(-\eta)\phi(a^{-1}))\omega(X)]_0) = 0 \qquad \forall X \in Q_{u_o h};$$

that is

$$\operatorname{tr}(a^{-1}\omega_0(X)a - [\eta, a^{-1}\omega_{-1}(X)]) = 0 \qquad \forall X \in Q_{u_o h};$$

hence

$$\langle \eta, a^{-1}\omega_{-1}(X)\rangle = 0 \qquad \forall X \in Q_{u_o h}$$

and since $\omega_{-1}|_{u_o h}: Q_{u_o h} \to \mathbf{R}^n$ is surjective and $a \in \mathbf{GL}(n, \mathbf{R})$, we have that $\eta = 0$, so that $h^{-1}h' \in \mathbf{GL}(n, \mathbf{R})$.

So we can define a differentiable map λ

$$\lambda: P \to H/\mathbf{GL}(n, \mathbf{R})$$

where for $u \in P$ we put $\lambda(u)[h]$, $[h]$ being the class in $H/\mathbf{GL}(n, \mathbf{R})$ of an element $h \in H$ with

$$\operatorname{tr} \omega_0|_{uh}(X) = 0 \qquad \forall X \in Q_{uh}.$$

Moreover it is quite elementary to check that the map λ is H-equivariant, that is

$$\lambda(uh) = h^{-1}\lambda(u) \qquad \forall u \in P, \quad \forall h \in H.$$

Our aim is now to prove that $\Lambda = \lambda^{-1}([Id])$ is a $\mathbf{GL}(n, \mathbf{R})$-principal subbundle of P; if we prove this, then

$$\Lambda = \{u \in P | \operatorname{tr} \omega_0(X) = 0 \ \forall X \in Q_u\}$$

is G-invariant, since G preserves both ω and Q; hence it yields a G-invariant subbundle of P that is isomorphic to $P^1(M)$ and the theorem is proved. To prove our claim, we establish a more general result, that is a generalization of a reduction theorem due to Singer ([6], p. 690):

Lemma 2.1. *Let $(P(M, G), M, \pi)$ be a principal fibre bundle over M with structural group G. Suppose now H to be a closed subgroup of G and suppose there is a differentiable map Ψ*

$$\Psi: P(M, G) \to G/H$$

that is G-equivariant, that is

$$\Psi(ug) = g^{-1}\Psi(u) \qquad \forall u \in P(M, G), \quad \forall g \in G.$$

Then $C = \Psi^{-1}([1_G])$ with projection map $\pi|_C$ is a principal fibre bundle over M with structural map H.

Proof. We construct a section σ of the bundle $P(M, G)/H$ over M in the following way: We note that for every $u \in P(M, G)$ the point $u\Psi(u) \in P(M, G)/H$ depends only on $\pi(u)$. Indeed if $g \in G$ we have that $ug\Psi(ug) = u\Psi(u)$; hence we can put $\sigma(\pi(u)) = u\Psi(u)$ and we get a differentiable section over M of $P(M, G)/H$. Now there is a one-to-one correspondence between the sections $\Gamma(M, P(M, G)/H)$ and the subbundles of $P(M, G)$ with structure group H (see [4]): if σ is a section then

$$Q_\sigma = \{u \in P(M, G) | \sigma(u) = [u]_H\}$$

is the corresponding subbundle; in our case we have that

$$Q_\sigma = \{u \in P(M, G) | \Psi(u) = [1_G]\} = C$$

and we are done. Q.E.D.

We now turn back to the proof of Theorem 2.2.

We observe that since Λ is G-invariant, $Q_u \subset T\Lambda_u$ $\forall u \in \Lambda$. Moreover the invariant connection on $P^1(M)$ is given by the distribution $\Gamma_u = \{X \in T\Lambda | \omega_0(X) = 0\}$, while the distribution Q corresponds to the canonical connection on G/K (see [4], vol. I). We now show that the connection Γ satisfies the property $(*)$.

Let $u_o \in \Lambda$ be a point with $\pi(u_o) = p(1_G) = o$ and let X, Y be two vector fields on M; we denote by $\widetilde{X}$, $\widetilde{Y}$ and X', Y' the horizontal lifts of X, Y to Λ with respect to the connections Γ and Q, respectively. Since the restriction of ω_{-1} to Λ is the canonical 1-form θ of $P^1(M) \cong \Lambda$, we have that (see [4], vol. I)

$$\begin{aligned} \nabla_X Y - D_X Y &= u_o(\widetilde{X}(\theta(\widetilde{Y})) - X'(\theta(Y'))) \\ &= u_o((\widetilde{X} - X')(\theta(\widetilde{Y}))) \end{aligned}$$

because $\omega_{-1}(\widetilde{Y}) = \omega_{-1}(Y')$. Now $(\widetilde{X} - X')_{u_o} = A^*_{u_o}$ for some $A \in \mathbf{gl}(n, \mathbf{R})$ and since $\omega_0(\widetilde{X}) = 0$ and $\operatorname{tr}\omega_0(X') = 0$, we have that $\operatorname{tr} A = 0$. So if we extend $A^*_{u_o}$ by means of A^* to a vector field on Λ, by the structure equation we get

$$A^*(\theta(\widetilde{Y})) - \widetilde{Y}\omega_{-1}(A^*) + \omega_{-1}([A^*, \widetilde{Y}]) - [\omega(A^*), \omega(\widetilde{Y})]_{-1};$$

but $\omega_{-1}(A^*) = 0$ and $[A^*, \widetilde{Y}] = 0$, so that

$$A^*_{u_0}(\theta(\widetilde{Y})) = -[A, \omega_{-1}(\widetilde{Y})]_{u_o}$$

and our claim follows from the condition $\operatorname{tr} A = 0$. The uniqueness of such a connection is quite elementary, since it follows directly from (1.1).

We are now going to prove the second assertion. Let us consider $X_1, X_2 \in Q_{u_o}$, say $X_i = \Phi_{u_o^*}(Y_i)$, $Y_i \in \mathbf{m}$ for $i = 1, 2$. We extend X_i by means of Y_i^*, where Y_i^* denotes the vector field induced on P by the one-parameter subgroup generated by Y_i. Then since G preserves ω, we have that

$$Y_i^*\omega(Y_j^*) = \omega([Y_i^*, Y_j^*]) = -\omega([Y_i, Y_j]^*) \qquad \forall i = 1, 2$$

and so

$$d\omega(X_1, X_2) = -\tfrac{1}{2}\omega([Y_1, Y_2]^*_{u_o}).$$

From the structure equation for ω we obtain

$$\operatorname{tr} d\omega_0(X_1, X_2) = -\tfrac{1}{2}\operatorname{tr}([\omega(X_1), \omega(X_2)]_0) = -\tfrac{1}{2}\operatorname{tr}\omega_0([Y_1 Y_2]^*_{u_o}).$$

Hence, if $[Y_1, Y_2]_{\mathbf{k}}$ denotes the component of $[Y_1, Y_2]$ along $\mathbf{k}$ with respect to the decomposition of $\mathbf{g}$,

$$\begin{aligned}(n+1)[\langle\omega_1(X_2), \omega_{-1}(X_1)\rangle - \langle\omega_1(X_1), \omega_{-1}(X_2)\rangle] &= \operatorname{tr}\omega_0([Y_1, Y_2]^*_{u_o}) \\ &= \operatorname{tr}\omega_0([Y_1, Y_2]^*_{\mathbf{k}^{u_o}})\end{aligned} \tag{2.4}$$

since $\operatorname{tr}\omega_0(X) = 0$ $\forall X \in Q_{u_o}$. If $Z_i = \pi_*(X_i)$ $(i = 1, 2)$ and if R is the curvature tensor for the canonical connection of G/K, then (see [4], vol. II)

$$\operatorname{tr}\omega_0([Y_1, Y_2]^*_{\mathbf{k}^{u_o}}) = -\operatorname{tr} R_o(Z_1, Z_2). \tag{2.5}$$

We note that for every $i = 1, 2$ there exists a unique $\widetilde{X}_i \in \Gamma_{u_o}$ with $\pi_*(\widetilde{X}_i) = \pi_*(X_i)$; it follows that

$$\omega_a(\widetilde{X}_i) = \omega_a(X_i)$$

for $a = -1, 1$ and $i = 1, 2$, since $\widetilde{X}_i - X_i$ is tangent to the fibre of Λ through u_o.

Now we use a formula established by Tanaka ([7], formula (3.1); the Ricci tensor is taken there with opposite sign):

$$\langle\omega_1(\widetilde{X}_i), \omega_{-1}(\widetilde{X}_j)\rangle = -\frac{1}{n^2-1}(\mathrm{Ric}_o(Z_j, Z_i) + n\mathrm{Ric}_o(Z_i, Z_j)) \tag{2.6}$$

where Ric is the Ricci tensor for the connection Γ.

Hence from (2.4)–(2.6) we have that

$$2A_o(Z_1, Z_2) = \operatorname{tr} R_o(Z_1, Z_2) \tag{2.7}$$

where A denotes the skew-symmetric part of Ric. Q.E.D.

Remark I. If $K = \{1\}$, then the canonical connection is flat; hence there exists an invariant connection with symmetric Ricci tensor. The same conclusion holds if the homogeneous space G/K admits a G-invariant volume form v: in this case v is parallel with respect to the canonical connection of G/K and this forces $\operatorname{tr} R = 0$ (for this remark see also [5]).

Remark II. If the Lie algebra $\mathbf{g}$ of G is semisimple, then there is at most one invariant connection belonging to $\mathscr{P}$ and with symmetric Ricci tensor: indeed two such connections differ by a G-invariant closed 1-form α (see [2]), so that $p^*\alpha$ is also closed and left invariant on G; this means that $\mathscr{A} = \ker \alpha_{1_G}$ is an ideal of $\mathbf{g}$ of codimension ≤ 1, hence $[\mathbf{g}, \mathbf{g}] \subset \mathscr{A}$; since $\mathbf{g}$ is semisimple we have $[\mathbf{g}, \mathbf{g}] = \mathbf{g}$ and $\mathscr{A} = \mathbf{g}$.

Corollary 2.1. *Let G/K be a reductive homogeneous space and let $\mathbf{g} = \mathbf{k} \oplus \mathbf{m}$ be a decomposition as in* (2.1). *There is a one-to-one correspondence between G-invariant projective structures on G/K and bilinear symmetric* $\operatorname{ad}(K)$*-invariant maps*

$$T: \mathbf{m} \times \mathbf{m} \to \mathbf{m}$$

such that

$$\operatorname{tr} T_X = 0 \qquad \forall X \in \mathbf{m}$$

where $T_X(Y) = T(X, Y)\ \forall X, Y \in \mathbf{m}$.

Proof. We denote by D and $\tilde{D}$ the canonical and the canonical torsionfree connection on G/K with respect to the given decomposition. Then to each G-invariant projective structure we can associate the G-invariant torsionfree connection ∇ given by Theorem 2.2. If S is the torsion tensor of D, we define the 1-form ϕ at each $p \in M = G/K$ as follows:

$$\phi_p(X) = \frac{1}{n+1} \operatorname{tr} \left\{ TM_p \ni Y \mapsto \frac{1}{2} S_{p_X} Y + (D_X - \nabla_X)_p Y \right\}$$

for each $X \in TM_p$ (n is the dimension of M). Then it is clear that if we take $\tilde{\nabla}$ projectively related to ∇ by means of the G-invariant 1-form ϕ, then the difference tensor $T = \tilde{D} - \tilde{\nabla}$ is symmetric and G-invariant; if we identify $\mathbf{m}$ with TM_o with $o = \pi(1_G)$ and $\pi: G \to M$ the canonical projection, we get our desired bijection. Q.E.D.

Remark III. By Corollary 2.1 we have that on the sphere $S^n = SO(n+1)/SO(n)$ and on the hyperbolic space $H^n = SO(1, n)/SO(n)$ the only invariant projective structure is that induced by the canonical connection: indeed the isotropy action of $SO(n)$ on $TM_o \equiv \mathbf{R}^n$ is the usual one and on $\mathbf{R}^n$ there is no $(1, 2)$ symmetric tensor invariant under the action of $SO(n)$ other than the null tensor.

Corollary 2.2. *Let $M = G/K$ be a symmetric space of dimension n with the canonical decomposition of the Lie algebra of G and let σ be the symmetry at $o = \pi(1_G)$, where $\pi: G \to M$ is the projection.*

(a) *If* $\mathscr{P}$ *is a projective structure on* M *which is invariant by the action of* G *and* σ, *then* $\mathscr{P}$ *is the projective structure induced by the canonical connection of* G/K.

(b) *If moreover the Lie algebra of the linear holonomy group with respect to the canonical connection* D *of* G/K *is contained in* $\mathbf{sl}(n)$, *then there is a bijection between flat G-invariant projective structures on* M *and symmetric bilinear* $\mathrm{ad}(K)$*-invariant maps as in Corollary* 2.1 *such that*

(**)
$$W_{oXY}Z = [T_X, T_Y]Z + \frac{1}{n-1}[\mathrm{tr}(T_Y T_Z)X - \mathrm{tr}(T_X T_Z)Y] \qquad \forall X, Y, Z \in \mathbf{m}$$

where W_o *is the Weyl curvature tensor at* o *of the canonical connection.*

Proof. (a) By Theorem 2.1 we know of the existence of a torsionfree invariant connection ∇ belonging to $\mathscr{P}$ and such that condition (*) is fulfilled. If D denotes the canonical connection for G/K, and if J is the tensor given by $J = \sigma^*\nabla - D$, then $J = \sigma^*(\nabla - D)$, since D is σ-invariant; hence for every $X \in TM_o$
$$\mathrm{tr}\{TM_o \ni Y \mapsto J_X Y\} = 0.$$
Since $\sigma^*\nabla$ fulfills property (*) and is projectively related to ∇ by hypothesis, we get $\sigma^*\nabla = \nabla$; but D is the unique G-invariant connection on M that is σ-invariant, so that $\nabla = D$ and we are done.

(b) We recall that D is torsionfree. If $\mathscr{P}$ is a G-invariant projective structure, we can pick the invariant connection ∇ belonging to $\mathscr{P}$ as in Theorem 2.2. If T is the difference tensor $D - \nabla$, then T is symmetric and D-parallel (since it is G-invariant); hence if R^* and R are the curvature tensors of D and ∇, respectively, we have
$$R^*_{XY} = R_{XY} + [T_X, T_Y].$$
By our assumptions $\mathrm{tr}\, R^*_{XY} = 0$, hence D and ∇ have symmetric Ricci tensors and we get
$$\mathrm{Ric}^*_{YZ} = \mathrm{Ric}_{YZ} - \mathrm{tr}(T_Y T_Z).$$
Since the Weyl curvature tensor is given in this case (see [2]) by
$$W_{XY}Z = R_{XY}Z - \frac{1}{n-1}[\mathrm{Ric}_{YZ}X - \mathrm{Ric}_{XZ}Y]$$
we have that $\mathscr{P}$ is flat iff (**) holds. Q.E.D.

Corollary 2.3. *If* G *is a Lie group with a projective structure* $\mathscr{P}$ *that is invariant under all left and right translations of* G, *then there exists a biinvariant affine connection on* G *that belongs to* $\mathscr{P}$ *and has symmetric Ricci tensor.*

Proof. (See also [5].) The group $G \times G = K$ acts on G in the following way:
$$G \times G \times G \to G,$$
$$((g, h), k)) \mapsto gkh^{-1},$$

and G turns out to be a reductive homogeneous space $G = K/\Delta G$ where ΔG is the diagonal of K ; our claim follows from the fact that K leaves the projective structure $\mathscr{P}$ invariant. Q.E.D.

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