AN ANALOGUE TO GLAUBERMAN'S ZJ-THEOREM

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ABSTRACT. Let $P$ be a finite $p$-group, $p$ an odd prime. Using certain versions of $p$-stability it is shown that there exists a nontrivial characteristic subgroup $W$ in $P$ that is normal in every finite $p$-stable group $G$ satisfying $C_G(O_p(G)) \leq O_p(G)$ and $P \leq \text{Syl}_p(G)$. Moreover, $W$ contains every abelian subgroup of $P$ normalized by $W$.

Let $p$ be an odd prime and $H \neq 1$ a finite $p$-stable group (for a definition of $p$-stability see [3]) with $C_H(O_p(H)) \leq O_p(H)$. In [1] Glauberman has proven:

ZJ-Theorem. Suppose that $S \in \text{Syl}_p(H)$, then $Z(J(S))$ is normal in $H$.

His theorem shows that there exists a fixed nontrivial characteristic subgroup of $S$, namely $Z(J(S))$, which is normal in every finite group $H$ containing $S$ as a Sylow $p$-subgroup and satisfying the above hypotheses. In this paper we want to prove an analogue to his theorem that does not use $Z(J(S))$.

Let $p$ be a prime and $S$ be a finite $p$-group. An embedding of $S$ is a pair $(\tau, H)$ where $H$ is a group and $\tau$ is a monomorphism from $S$ into $H$. Two embeddings $(\tau_1, H_1)$ and $(\tau_2, H_2)$ are equivalent, if there exists an isomorphism $\phi$ from $H_1$ onto $H_2$ so that $\tau_1 \phi = \tau_2$. It is easy to see that this defines an equivalence relation on the class of all embeddings of $S$.

In the following two lemmas let $\mathcal{E}$ be a set of embeddings of $S$. By $W_\mathcal{E}(S)$ we denote the largest subgroup of $S$ so that $W_\mathcal{E}(S)\tau$ is normal in $H$ for every $(\tau, H) \in \mathcal{E}$.

Lemma 1. Let $\alpha \in \text{Aut}(S)$. Suppose that $(\alpha \tau, H)$ is equivalent to an element in $\mathcal{E}$ for every $(\tau, H) \in \mathcal{E}$. Then $W_\mathcal{E}(S)$ is $\alpha$-invariant.

Proof. Let $(\alpha \tau, H)$ be equivalent to $(\tilde{\tau}, \tilde{H}) \in \mathcal{E}$, i.e., there exists an isomorphism $\phi:\tilde{H} \rightarrow H$ so that $\alpha \tau = \tilde{\tau} \phi$. It follows that $W_\mathcal{E}(S)\alpha \tau = W_\mathcal{E}(S)\tilde{\tau} \phi$, and so $(W_\mathcal{E}(S)\alpha)\tau$ is normal in $H$. Since this holds for every $(\tau, H) \in \mathcal{E}$ we get $W_\mathcal{E}(S)\alpha = W_\mathcal{E}(S)$.

Lemma 2. Let $(\tau, H) \in \mathcal{E}$ and $\beta \in \text{Aut}(H)$. Then $(\tau, H)$ is equivalent to $(\tau \beta, H)$.
The proof is obvious.
In this paper the following two versions of \( p \)-stability will be used.

**Definition.** Let \( X \) be a finite group and \( p \) an odd prime so that \( C_X(O_p(X)) \leq O_p(X) \). Set \( Z_0(N) = Z(O_p(O'_p(N))) \) for every normal subgroup \( N \) of \( X \).

**Weak \( p \)-stability.** \( X \) is weakly \( p \)-stable, if for every \( x \in X \)
\[ [V, x, x] = 1 \] implies \( xC_X(V) \in O_p(X/C_X(V)) \) for every abelian normal \( p \)-subgroup \( V \) of \( X \) and for \( V = O_p(V) \).

**Strong \( p \)-stability.** \( X \) is strongly \( p \)-stable, if \( N/Z_0(N) \) is weakly \( p \)-stable for every normal subgroup \( N \) of \( X \).

An embedding \((\tau, H)\) of \( S \) is weakly (strongly) \( p \)-stable, if \( H \) is weakly (strongly) \( p \)-stable.

**Lemma 3.** Let \( X \) be strongly \( p \)-stable and let \( N \) be a normal subgroup of \( X \). Then \( N/Y \) is weakly \( p \)-stable for every normal subgroup \( Y \) of \( N \) in \( Z_0(N) \).

**Proof.** Set \( \overline{N} = N/Z_0(N) \) and \( \overline{N} = N/Y \), and let \( V \) be a normal \( p \)-subgroup of \( N \) so that \( \overline{V} \) is abelian or \( \overline{V} = O_p(\overline{N}) \). Suppose that \( [V, y, y] \leq Y \) for some \( p \)-element \( y \) in \( N \). Since \( \overline{N} \) is weakly \( p \)-stable we get that \( [\overline{V}, O_p(\langle y^\overline{N} \rangle)] = 1 \) and thus also \( [\overline{V}, O_p(\langle y^\overline{N} \rangle)] = 1 \), i.e., \( \tilde{y} \in O_p(\tilde{N}/C_{\tilde{N}}(\tilde{V})) \).

**Remark.** The \( p \)-stability defined in [3] implies weak \( p \)-stability (for groups \( X \) satisfying \( C_X(O_p(X)) \leq O_p(X) \)). On the other hand, there are examples of weakly \( p \)-stable groups that are not \( p \)-stable.

**Lemma 4.** Let \( p \) be an odd prime and \( S \) be a \( p \)-group. Suppose that \( \mathcal{E} \) is a set of weakly \( p \)-stable embeddings of \( S \) so that \( S \tau \in Syl_p(H) \) for \((\tau, H) \in \mathcal{E}\). Then \( Z(S) \leq W_{\mathcal{E}}(S) \). In particular \( W_{\mathcal{E}}(S) \neq 1 \) if \( S \neq 1 \).

**Proof.** Let \( \mathcal{E} = \{ (\tau_i, H_i) \mid i \in I \} \) and let \( G \) be the amalgamated product of \( H_i \), \( i \in I \), over \( S \) (for the definition see [4]). As usual we identify the groups \( H_i \) and \( S \) with the corresponding subgroups of \( G \), i.e., \( G = \langle H_i \mid i \in I \rangle \) and \( S = \cap_{i \in I} H_i \). Then \( W_{\mathcal{E}}(S) \) is the largest subgroup of \( S \) that is normal in \( G \).

Let \( \Gamma \) be the coset graph of \( G \) with respect to the subgroups \( H_i \), \( i \in I \), and \( S \), i.e., the vertices of \( \Gamma \) are the cosets \( H_i x, i \in I \), and \( S x \) for \( x \in G \), and two vertices \( Ax \) and \( By \) are adjacent, if and only if \( Ax \neq By \) and \( Ax \subseteq By \) or \( By \subseteq Ax \). Then \( G \) operates on \( \Gamma \) by right multiplication. We identify \( \Gamma \) with its vertex set and use the following notation for \( \delta \in \Gamma \).

\[
\begin{align*}
  d(\cdot , \cdot ) & : \text{ the usual distance metric on } \Gamma , \\
  \Delta(\delta) & : \text{ the set of vertices adjacent to } \delta , \\
  G_\delta & : \text{ the stabilizer of } \delta \text{ in } G , \\
  Q_\delta & = O_p(G_\delta) , \\
  Z_\delta & = \langle Z(T) \mid T \in Syl_p(G_\delta) \rangle .
\end{align*}
\]

(1) \( O_p(G_\delta/C_{G_\delta}(Z_\delta)) = 1 \).

This follows from the definition of \( Z_\delta \) and the Frattini argument.
The next properties are immediate consequences of the definition of \( \Gamma \) (see also [2]).

(2) \( \Gamma \) is connected.

(3) \( G_\delta \) is conjugate in \( G \) to \( S \) or some \( H_i, i \in I \).

(4) \( G_\delta \cap G_\lambda \in \text{Syl}_p(G_\delta) \) for \( \lambda \in \Delta(\delta) \).

(5) \( Q_\delta = \bigcap_{\lambda \in \Delta(\delta)} (G_\delta \cap G_\lambda) \).

(6) \( Z_\delta < Z(Q_\delta) \).

(7) \( G_\delta \) is transitive on \( \Delta(\delta) \), or \( G_\delta \) is conjugate to \( S \) and \( G_\delta = Q_\delta \).

Since \( W_\delta(S) \) is the largest normal subgroup of \( G \) contained in \( S \) we conclude from (5) that \( W_\delta(S) \) is the kernel of the operation of \( G \) on \( \Gamma \).

We now assume that \( Z(S) \not\subseteq W_\delta(S) \). Then (5) yields:

(8) There exist \( \alpha, \alpha' \in \Gamma \) so that \( Z_\alpha \not\subseteq Q_{\alpha'} \).

Let \( \alpha \) and \( \alpha' \) be as in (8) so that \( b := d(\alpha, \alpha') \) is minimal, i.e., \( Z_\alpha \leq Q_\delta \) for every \( \delta \in \Gamma \) with \( d(\alpha, \delta) < b \); in particular \( Z_\alpha \leq G_{\alpha'} \) by (5).

Since every normal \( p \)-subgroup of \( G_{\alpha'} \) is in \( Q_{\alpha'} \) we have that \( \langle Z_{\alpha'} \rangle \) is non-abelian. We get:

(9) There exist \( \lambda, \lambda' \in \Gamma \) so that \( [Z_\lambda, Z_{\lambda'}] \neq 1 \).

Let \( \lambda \) and \( \lambda' \) be as in (9) so that \( r := d(\lambda, \lambda') \) is minimal, i.e., \( [Z_\lambda, Z_\delta] = 1 \) for every \( \delta \in \Gamma \) with \( d(\lambda, \delta) < r \). Let

\[ \Lambda_r = \{ (\lambda, \lambda') \mid d(\lambda, \lambda') = r \text{ and } [Z_\lambda, Z_{\lambda'}] \neq 1 \} \]

(10) \( b > r/2 \).

Assume that \( b \leq r/2 \). Note that \( [Q_{\alpha'}, Z_\alpha] \leq \langle Z_{Q_{\alpha'}} \rangle \) and that by (5) \( d(\alpha, \alpha^q) < r \) for \( q \in Q_{\alpha'} \). It follows from the minimality of \( r \) that \( \langle Z_{Q_{\alpha'}} \rangle \) is abelian, i.e., \( [Q_{\alpha'}, Z_\alpha, Z_\beta] = 1 \). Since \( G_{\alpha'} \) is weakly \( p \)-stable by (3) and \( C_{G_{\alpha'}}(Q_{\alpha'}) \subseteq Q_{\alpha'} \) we get \( Z_\alpha \leq Q_{\alpha'} \), a contradiction.

Now let \( (\lambda, \lambda') \in \Lambda_r \) and \( (\lambda_0, \ldots, \lambda_r) \) be a path of length \( r \) with \( \lambda_0 = \lambda \) and \( \lambda_r = \lambda' \). Note that \( Z_{\lambda_i} \leq G_{\lambda_i} \) for \( i \leq b \).

(11) Let \( b \leq s \) and \( V = \langle Z_{\lambda'} \rangle \). Suppose that \( Z_{\lambda_i} \leq G_{\lambda_i} \) for \( 0 \leq i \leq s \).

Then the following hold:

(a) \( V \) is an abelian normal subgroup of \( G_{\lambda_s} \) and \( [V, Z_{\lambda_s}, Z_{\lambda_s}] = 1 \).

(b) If \( s \neq r \), then there exists \( (\lambda, \lambda'') \in \Lambda_r \) and a path \( (\lambda_0, \ldots, \lambda_r) \) of length \( r \) with \( \lambda_0 = \lambda \) and \( \lambda_r = \lambda'' \) and \( Z_{\lambda_i} \leq G_{\lambda_i} \) for \( 0 \leq i \leq s + 1 \).

Note that \( Z_{\lambda'} \) is abelian by (6) and \( d(\lambda_i, \lambda') \leq r - b < r/2 < b \) by (10). Hence \( Z_{\lambda'} \leq Q_{\lambda_j} \) and \( d(\lambda_i', \lambda_j') < r \) for \( g \in G_{\lambda_j} \), and \( V \) is abelian.

Now let \( W = \langle Z_{\lambda'} \rangle \). Since \( V \) fixes \( \lambda_{r-b} \), we get \( d(\lambda, \lambda'') \leq 2(r-b) < r \) for \( v \in V \), and \( W \) is abelian. This implies \( [V, Z_{\lambda_s}, Z_{\lambda_s}] = 1 \), and (a) holds.

Assume now that \( s = r \). Let \( N \) be the inverse image of \( O_p(G_{\lambda_s}/C_{G_{\lambda_s}}(V)) \) in \( G_{\lambda_s} \). Since \( G_{\lambda_s} \) is weakly \( p \)-stable by (3) we get from (a) that \( Z_{\lambda_s} \leq N \).
Let \( T = G_{\lambda_{s+1}} \cap N \) and \( \tilde{T} = G_{\lambda_1} \cap N \). By (4) \( T, \tilde{T} \in \text{Syl}_p(N) \) and \( Z_{\lambda} \leq T \).

Moreover, we have \( N = TC_N(V) \). Hence, there is \( c \in C_N(V) \) so that \( \tilde{T}^c = T \), i.e., \( Z_{\lambda} \leq G_{\lambda_{s+1}} \). Since \( Z_{\lambda'} = Z_{\lambda'}^c = Z_{\lambda'} \), claim (b) follows for \( \lambda'' = \lambda' \) and the path \( (\lambda_0, \ldots, \lambda_r, \lambda, \lambda_{s+1}, \ldots, \lambda_s) \).

We now derive a final contradiction. According to (11)(b) we can choose

\((\lambda, \lambda') \in \Lambda_r \) and \( (\lambda_0, \ldots, \lambda_r) \) so that \( \lambda_0 = \lambda, \lambda_r = \lambda' \) and \( Z_{\lambda} \leq G_{\lambda_i} \) for \( i = 0, \ldots, r \). Now we get from (11)(a) for \( s = r \) that \( [Z_{\lambda'}, Z_{\lambda}, Z_{\lambda}] = 1 \).

Since \( G_{\lambda'} \) is weakly \( p \)-stable we conclude

\[ Z_{\lambda}C_{G_{\lambda'}}(Z_{\lambda'})/C_{G_{\lambda'}}(Z_{\lambda'}) \leq O_p(G_{\lambda'}/C_{G_{\lambda'}}(Z_{\lambda'})). \]

But now (1) implies \( Z_{\lambda} \leq C_{G_{\lambda'}}(Z_{\lambda'}), \) which contradicts the definition of \( \Lambda_r \).

**Theorem 1.** Let \( p \) be an odd prime and \( S \) a \( p \)-group. Then there exists a characteristic subgroup \( W(S) \) of \( S \) so that \( Z(S) \leq W(S) \), and \( W(S) \) is normal in every weakly \( p \)-stable group \( H \) with \( S \in \text{Syl}_p(H) \).

**Proof.** Let \( \mathcal{E} \) be a maximal set of pairwise nonequivalent weakly \( p \)-stable embeddings of \( S \) so that \( S\tau \in \text{Syl}_p(H) \) for \( (\tau, H) \in \mathcal{E} \). Note that \( \mathcal{E} \) exists since by condition \( C_H(O_p(H)) \leq O_p(H) \) there are only finitely many nonequivalent weakly \( p \)-stable embeddings \( (\tilde{\tau}, \tilde{H}) \) of \( S \) with \( S\tilde{\tau} \in \text{Syl}_p(\tilde{H}) \). Moreover, every weakly \( p \)-stable embedding \( (\tilde{\tau}, \tilde{H}) \) of \( S \) with \( S\tilde{\tau} \in \text{Syl}_p(\tilde{H}) \) is equivalent to some element in \( \mathcal{E} \). Hence, by Lemma 1 \( W_\mathcal{E}(S) \) is a characteristic subgroup of \( S \) and by Lemma 4 \( Z(S) \leq W_\mathcal{E}(S) \). Thus, Theorem 1 holds for \( W(S) := W_\mathcal{E}(S) \).

In the next theorem we denote by \( W(S) \) the largest characteristic subgroup of \( S \) for which Theorem 1 holds.

**Theorem 2.** Let \( p \) be an odd prime and \( S \) a \( p \)-group. Then there exists a characteristic subgroup \( W^*(S) \) of \( S \) so that

(i) \( Z(S) \leq W(S) \leq W^*(S) \),

(ii) \( W^*(S) \) is normal in every strongly \( p \)-stable group \( H \) with \( S \in \text{Syl}_p(H) \),

(iii) \( C_S(W^*(S)) \leq W^*(S) \).

(iv) If \( A \leq S \) and \( [W^*(S), A, A] = 1 \), then \( A \leq W^*(S) \). In particular, every normal abelian subgroup of \( S \) is contained in \( W^*(S) \).

**Proof.** Let \( \mathcal{E} \) be a maximal set of pairwise nonequivalent strongly \( p \)-stable embeddings of \( S \) so that \( S\tau \in \text{Syl}_p(H) \) for \( (\tau, H) \in \mathcal{E} \). As in Theorem 1 \( \mathcal{E} \) is finite and every strongly \( p \)-stable embedding \( (\tilde{\tau}, \tilde{H}) \) of \( S \) with \( S\tilde{\tau} \in \text{Syl}_p(\tilde{H}) \) is equivalent to some element in \( \mathcal{E} \).

Set \( W^*(S) = W_\mathcal{E}(S) \). Note that \( H \) is weakly \( p \)-stable for every \( (\tau, H) \in \mathcal{E} \).

Hence, \( W^*(S) \) is characteristic subgroup of \( S \) by Lemma 1, and Theorem 1 gives (i) and (ii).

For \( (\tau, H) \in \mathcal{E} \) we define:

\[ S_1 = C_S(W^*(S)), \quad H_1 = C_H(W^*(S)), \quad \tau_1 = \tau|_{S_1}. \]
Then $E_1 = \{(\tau_1, H_1) \mid (\tau, H) \in E\}$ is a set of embeddings of $S_1$, and since $H_1$ is normal in $H$ and $H$ is strongly $p$-stable, all these embeddings are weakly $p$-stable by Lemma 3.

Set $W_*(S_1) = W_{E_1}(S_1)$. By Lemma 4 we have $Z(S_1) \leq W_*(S_1)$. Moreover, $H = H_1 N_H(S_1 \tau_1)$ for $(\tau_1, H_1) \in E_1$, and Lemma 2 together with Lemma 1 shows that $W_*(S_1) \tau_1$ is normal in $H$. By the maximality of $W^*(S)$ we get

$$W_*(S_1) \leq W^*(S) \quad \text{and} \quad W_*(S_1) = Z(S_1);$$

in particular $Z(S_1) \tau_1 = Z(H_1)$ for $(\tau_1, H_1) \in E_1$.

For $(\tau_1, H_1) \in E_1$ we now define:

$$S_1 = S_1 / Z(S_1) \quad \text{and} \quad H_1 = H_1 / Z(S_1) \tau_1,$$

and $\tau_1$ is the monomorphism into $H_1$ induced by $\tau_1$. Then $E_1 = \{(\tau_1, H_1) \mid (\tau_1, H_1) \in E_1\}$ is a set of embeddings of $S_1$, and all these embeddings are weakly $p$-stable by Lemma 3.

Set $W_*(S_1) = W_{E_1}(S_1)$ and let $\tilde{W}_*(S_1)$ be the inverse image of $W_*(\bar{S}_1)$ in $S_1$. Note that by Lemma 4 $Z(\bar{S}_1) \leq W_*(\bar{S}_1)$. It follows that $\tilde{W}_*(S_1) \tau_1$ is normal in $H_1$ for $(\tau_1, H_1) \in E_1$ and thus $Z(S_1) \leq \tilde{W}_*(S_1) \leq W_*(S_1) = Z(S_1)$, i.e.,

$$W_*(S_1) = \tilde{W}_*(S_1) \quad \text{and} \quad Z(\bar{S}_1) = W_*(\bar{S}_1) = 1.$$

This implies

$$W_*(S_1) = Z(S_1) = S_1 = H_1$$

and $C_S(W^*(S)) = S_1 = W_*(S_1) \leq W^*(S)$. This proves (iii).

Suppose that $[W^*(S), A, A] = 1$, but $A \notin W^*(S)$ for some $A \leq S$. As in the proof of Lemma 4 let $G$ be the amalgamated product of the $H_i$'s over $S$ where $(\tau_i, H_i) \in E$. Then $W^*(S)$ is the kernel of the operation of $G$ on the corresponding coset graph. Since $A \notin W^*(S)$ there exists a $G$-conjugate $A^*$ of $A$ in $S$ and $(\tau, H) \in E$ so that $A^* \notin O_p(H)$.

Set $H_0 = (A^*H)$. Then strong $p$-stability gives $[W^*(S), O^p(H_0)] = 1$, and (iii) implies that $O^p(H_0) = 1$, i.e., $A^* \leq O_p(H)$, a contradiction.

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References


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