

ON $(0, 1, 2)$ INTERPOLATION IN UNIFORM METRIC

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ABSTRACT. From the well known theorem of G. Faber it follows that for any given matrix of nodes there exists a continuous function for which the Lagrange interpolation polynomial $L_n[f, x]$, generated by the n th row of the matrix, does not tend uniformly to $f(x)$. In this paper we shall provide analogous results for the related operator $H_{n,3}[f, x]$ as defined below.

Let

$$(1) \quad (-1 \leq) x_1 < x_2 < \cdots < x_n (\leq 1)$$

be an arbitrary system of nodes of interpolation ($x_k = x_{kn}$, $k = 1, \dots, n$; $n = 1, 2, \dots$), and for an arbitrary continuous function $f(x)$ in $[-1, 1]$ (i.e., $f \in C[1, 1]$) and integer $m \geq 1$, consider the $(0, 1, \dots, m)$ Hermite-Fejér interpolation of order m defined by

$$H_{n,m}^{(j)}(f, x_k) = \delta_{0,j} f(x_k) \quad (k = 1, \dots, n; j = 0, \dots, m-1).$$

Evidently, $H_{n,m}(f, x)$ is a uniquely determined polynomial of degree at most $mn - 1$.

$H_{n,1}(f, x)$ is the Lagrange interpolation polynomial of $f(x)$; the classical result of G. Faber [4] shows that this cannot be uniformly convergent for all $f \in C[-1, 1]$ for any system of nodes (1); while another classical result of P. Erdős and P. Turán [2] asserts that if (1) are the roots of the n th orthogonal polynomial with respect to an arbitrary L^1 -integrable weight function $w(x) \geq 0$, then $H_{n,1}(f, x)$ converges in weighted L^2 metric for any $f \in C[-1, 1]$.

For $m = 2$, the situation is different. There exist systems of nodes (1) such that $H_{n,2}(f, x)$ uniformly converges for all $f \in C[-1, 1]$ (e.g., for the roots of the Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ with $-1 < \alpha, \beta < 0$; see G. Szegő [6], Theorem 14.6). Hence in this case the L^2 convergence follows automatically and is of no interest.

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For $m = 3$ the results are less complete. R. Sakai [5] proved that for the Chebyshev roots $H_{n,3}(f, x)$ cannot converge for all $f \in C[-1, 1]$ (actually he proved this for all odd m 's), and later P. Vértesi [7, Theorem 2.7] generalized this result for arbitrary Jacobi nodes (under some additional condition whose validity is checked only for $m \leq 5$). Our result here is that for any system of nodes (1), $H_{n,3}(f, x)$ cannot converge uniformly for all $f \in C[-1, 1]$. This follows from the following more quantitative result on the norm of $H_{n,3}$.

Theorem. For any system of nodes (1) we have

$$(2) \quad |||H_{n,3}||| \geq c \log n.$$

Proof. An easy calculation shows that

$$(3) \quad H_{n,3}(f, x) = \sum_{k=1}^n f(x_k) A_k(x),$$

where

$$(4) \quad A_k(x) = \{1 - 3l'_k(x_k)(x - x_k) + [6l'_k(x_k)^2 - \frac{3}{2}l''_k(x_k)](x - x_k)^2\} l_k(x)^3$$

$(k = 1, \dots, n)$

with the usual notation $l_k(x)$ for the k th fundamental polynomial of Lagrange interpolation. Hence

$$|||H_{n,3}||| = \left\| \sum_{k=1}^n |A_k(x)| \right\|$$

where $\|\cdot\|$ is the supremum norm of the corresponding function. Since $1 - 3x + 4.5x^2 > 0$ for any real x , we get from (4)

$$1 - 3l'_k(x_k)(x - x_k) + \left[6l'_k(x_k)^2 - \frac{3}{2}l''_k(x_k)\right](x - x_k)^2 \geq \frac{3}{2} \left[l'_k(x_k)^2 - l''_k(x_k)\right](x - x_k)^2.$$

Here

$$l'_k(x_k)^2 - l''_k(x_k) = \left(\sum_{i \neq k} \frac{1}{x_k - x_i} \right)^2 - \sum_{\substack{i, j \neq k \\ i \neq j}} \frac{1}{(x_k - x_i)(x_k - x_j)} = \sum_{i \neq k} \frac{1}{(x_k - x_i)^2},$$

i.e.,

$$(5) \quad |A_k(x)| \geq \frac{3}{2} \frac{(x - x_k)^2}{(x_k - x_{k-1})^2} |l_k(x)|^3 \quad (2 \leq k \leq n).$$

In the rest of the proof of Theorem 1 we use a modification of the idea of proof of Theorem 1 in P. Erdős and P. Turán [3]. We distinguish two cases.

Case 1. There is a k_0 , $1 \leq k_0 \leq n$ such that

$$|l_{k_0}(\xi_0)| = \|l_{k_0}\| \geq n^2 \quad (\xi_0 \in [-1, 1]).$$

Then by Markov's theorem

$$(6) \quad |l_{k_0}(x)| \geq \frac{1}{2} \|l_{k_0}\| \geq \frac{1}{2} n^2 \left(|x - \xi_0| \leq \frac{1}{2n^2} \right).$$

Thus choosing a $\xi_1 \in [-1, 1]$ such that $|\xi_1 - x_{k_0}| \geq \frac{1}{2n^2} \geq |\xi_1 - \xi_0|$ we obtain from (5) and (6)

$$|A_{k_0}(\xi_1)| \geq \frac{3}{2} \cdot \frac{1}{4} \cdot \left(\frac{1}{2n^2}\right)^2 \left(\frac{1}{2}n^2\right)^3 = \frac{3}{2^8}n^2,$$

which is stronger than (2).

Case 2. $\|l_k\| < n^2$ ($k = 1, 2, \dots, n$). In this case, according to a result of P. Erdős [1], we have (with $x_k = \cos \theta_k$)

$$\left| \sum_{\theta_k \in I} 1 - \frac{|I|}{\pi} n \right| \leq \log^2 n \quad (I \subseteq [0, \pi]).$$

Thus

$$\sum_{\theta_k \in I} 1 \geq \frac{1}{15} |I| n \text{ if } |I| \geq 4 \frac{\log^2 n}{n}, \quad I \subseteq [0, \pi] \text{ and } n \geq n_0.$$

Therefore using the harmonic-geometric-arithmetic mean inequalities we get (7)

$$\begin{aligned} \sum_{\theta_k \in I} \frac{1}{(x_k - x_{k-1})^2} &\geq \frac{\sum_{\theta_k \in I} 1}{(\prod_{\theta_k \in I} (\theta_{k-1} - \theta_k))^{2/\sum_{\theta_k \in I} 1}} \geq \frac{(\sum_{\theta_k \in I} 1)^3}{[\sum_{\theta_k \in I} (\theta_{k-1} - \theta_k)]^2} \\ &\geq \frac{(\frac{1}{15}|I|n)^3}{|I|^2} > 10^{-4} |I| n^3 \text{ if } |I| \geq 4 \frac{\log^2 n}{n}, \quad I \subseteq [0, \pi], \quad n \geq n_0. \end{aligned}$$

Now let

$$(8) \quad \omega_n(x) = \prod_{k=1}^n (x - x_k), \quad I_n = \left[-\frac{1}{\log n}, \frac{1}{\log n}\right], \quad I'_n = \left[-\frac{1}{2 \log n}, \frac{1}{2 \log n}\right],$$

$$(9) \quad M_n = \max_{|x| \leq 1} |\omega_n(x)| = |\omega_n(\xi)|, \quad \overline{M}_n = \max_{x \in I'_n} |\omega_n(x)| = |\omega_n(\xi_0)| \quad (x = \cos \theta).$$

According to Lemma 1 in [3], we have

$$(10) \quad \max_{\theta \in I'_n} |\omega'_n(x)| = O(\eta_n n M_n), \quad \text{where } \eta_n = \max \left(\frac{1}{\log^2 n}, \frac{\overline{M}_n}{M_n} \right).$$

Subcase 2a. $\overline{M}_n \leq M_n / \log^2 n$. Then by (10)

$$\max_{\theta \in I'_n} |\omega'_n(x)| = O \left(\frac{n M_n}{\log^2 n} \right).$$

Choosing $I = I'_n$ in (7), we obtain from (5), (8), (9), and (7)

$$\begin{aligned} \sum_{\theta_k \in I'_n} |A_k(\xi)| &\geq \frac{3}{4} \sum_{\theta_k \in I'_n} \left| \frac{\omega_n(\xi)}{\omega'_n(x_k)} \right|^3 \frac{1}{(x_k - x_{k-1})^2} \\ &\geq c_1 \frac{\log^6 n}{n^3} \sum_{\theta_k \in I'_n} \frac{1}{(x_k - x_{k-1})^2} \geq c_2 \log^5 n. \end{aligned}$$

Subcase 2b. $\bar{M}_n > M_n/\log^2 n$. Then similarly as above,

$$\max_{\theta \in I'_n} |\omega'_n(x)| = O(n\bar{M}_n),$$

and

$$\begin{aligned} (11) \quad \sum_{\theta_k \in I'_n} |A_k(\xi_0)| &\geq \frac{3}{2} \sum_{\theta_k \in I'_n} \left| \frac{\omega_n(\xi_0)}{\omega'_n(x_k)} \right|^3 \frac{1}{|\xi_0 - x_k|(x_k - x_{k-1})^2} \\ &\geq \frac{c_3}{n^3} \sum_{\theta_k \in I'_n} \frac{1}{|\xi_0 - x_k|(x_k - x_{k-1})^2}. \end{aligned}$$

Let $\xi_0 = \cos \theta_0$ and $0 < \theta_0 \leq \frac{\pi}{2}$. Define

$$\begin{aligned} I_{n,\lambda} &= \left[\theta_0 + 4\lambda \frac{\log^2 n}{n}, \theta_0 + 4(\lambda + 1) \frac{\log^2 n}{n} \right] \subseteq I'_n \\ &\quad \left(\lambda = 0, 1, \dots, \left[\frac{n}{10 \log^3 n} \right] = m \right). \end{aligned}$$

Then by (7) (with $I = I_{n,\lambda}$)

$$\begin{aligned} \sum_{\theta_k \in I'_n} \frac{1}{|\xi_0 - x_k|(x_k - x_{k-1})^2} &\geq \sum_{\lambda=1}^m \sum_{\theta_k \in I_{n,\lambda}} \frac{1}{|\theta_0 - \theta_k|(x_k - x_{k-1})^2} \\ &\geq c_4 \frac{n}{\log^2 n} \sum_{\lambda=1}^m \frac{1}{\lambda} \sum_{\theta_k \in I_{n,\lambda}} \frac{1}{(x_k - x_{k-1})^2} \\ &\geq c_5 \frac{n}{\log^2 n} \cdot \log n \cdot \log^2 n \cdot n^2 \\ &= c_5 n^3 \log n, \end{aligned}$$

i.e., by (11)

$$\sum_{\theta_k \in I'_n} |A_k(\xi_0)| \geq c_3 c_5 \log n.$$

The theorem is completely proved.

Remark. We conjecture that the statement of our theorem remains true for any odd m .

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